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Certain Triply Orthogonal Systems of Surfaces.

By L. P. Eisenhart.

In a former paper* we have discussed the surfaces which have the same spherical representation of their lines of curvature as pseudospherical surfaces, and for convenience of reference have called them the A-surfaces. This paper is devoted to the study of triply orthogonal systems in which all the surfaces in one family are A-surfaces; we shall refer to them as A-systems.

In §1 we recall the general equations of condition for a triply orthogonal system, first found by Lame, and the formulas which give the expressions for the fundamental functions of the surfaces of a system. These are applied in §2 to the special case of A-systems, and it is found that there are three conditions to be satisfied by the spherical representation of the A-surfaces of the system, and three others by the coefficients of the linear element of space in terms of these surfaces. From the definition of A-surfaces it is seen that given a surface of this kind there is a pseudospherical surface with the same spherical representation of its lines of curvature; we shall say that the latter is the *associated* pseudospherical surface of the former. Given any A-system; the spherical representation of the associated pseudospherical surfaces evidently satisfies the three conditions mentioned above, and one finds that the coefficients of the linear element of space in terms of this new system satisfy the other three conditions identically; hence when one has a family of A-surfaces satisfying the conditions of Lame, the associated pseudospherical surfaces form a family of Lame. This shows that all A-systems can be obtained from pseudospherical systems by the transformation of Combescure.† A study of this transformation as applied to A-systems is made in §3, and several particular cases are discussed.

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*Surfaces with the same spherical representation of their lines of curvature as pseudospherical surfaces, American Journal, vol. 27, pp. 113–172.

†Bianchi, Lezioni, II, p. 490.

In the previous paper* we established certain transformations for *A*-surfaces which reduce in the case of pseudospherical surfaces to the well-known transformations of Bianchi and Bäcklund. In §4 we determine the conditions to be satisfied in order that a set of transforms of a Lame family of *A*-surfaces shall constitute such a family. It is found that when such a transformation is known for one system it can be applied to all systems with the same spherical representation. It is likewise shown that these transformations admit of a theorem of permutability reducing for pseudospherical systems to the theorem of Bianchi.†

Weingarten was the first to consider a pseudospherical system in which all the pseudospherical surfaces have the same total curvature. The representation upon the sphere takes a form of special interest for this case; it is considered in §5 and all *A*-systems with this representation are referred to as *Weingarten A-systems*. The moulure surfaces are *A*-surfaces, and §6 deals with systems in which all the surfaces in one system are of this kind; it is found that in such systems a second family also is composed of moulure surfaces.

In §7 we consider the *A*-systems for which the orthogonal trajectories of the family of *A*-surfaces are plane curves. It is shown that so far as the spherical representation is concerned this problem is reducible to the systems with circular trajectories; a large class of the latter are treated at length. One passes then to the consideration of *A*-systems in which one system of the lines of curvature on the *A*-surface are plane curves; this occupies §8. §9 deals with systems for which the trajectories of the *A*-surfaces have constant curvature; all of these systems are found to be pseudospherical.

In another place‡ we have considered at length the *A*-surfaces with spherical lines of curvature in one system. Now we study the *A*-systems in which all the *A*-surfaces are of this kind; the general discussion of this problem and the solution of several particular cases form the substance of §10. In §11 those *A*-systems are discussed in which the surfaces in one family are spheres.

We have shown before§ that certain surfaces discovered by Bianchi are *A*-surfaces. In §12 we find the conditions to be satisfied in order that all the surfaces in one family of a triple system are surfaces of one of the three types given by Bianchi, and we determine the forms of the linear element of space when such a system is parametric.

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*l. c., p. 148.

†l. c., p. 541.

‡Certain Surfaces with Plane or Spherical Lines of Curvature, Americal Journal, vol. 28, pp. 47–70.

§Surfaces analogous to the surfaces of Bianchi, Annali, Ser. III, vol. 12, pp. 113–143.

§1. *General Triply Orthogonal Systems of Surfaces.*

Consider space referred to a triply-orthogonal system of surfaces and write the linear element in the form

$$ds^2 = H_1^2 du_1^2 + H_2^2 du_2^2 + H_3^2 du_3^2. \tag{1}$$

Lame[*] has shown that the functions H_1, H_2, H_3 must satisfy six equations of condition, namely

$$\left\{\begin{aligned} \frac{\partial^2 H_1}{\partial u_2 \partial u_3} &= \frac{1}{H_2}\frac{\partial H_2}{\partial u_3}\frac{\partial H_1}{\partial u_2} + \frac{1}{H_3}\frac{\partial H_3}{\partial u_2}\frac{\partial H_1}{\partial u_3}, \\ \frac{\partial^2 H_2}{\partial u_3 \partial u_1} &= \frac{1}{H_3}\frac{\partial H_3}{\partial u_1}\frac{\partial H_2}{\partial u_3} + \frac{1}{H_1}\frac{\partial H_1}{\partial u_3}\frac{\partial H_2}{\partial u_1}, \\ \frac{\partial^2 H_3}{\partial u_1 \partial u_2} &= \frac{1}{H_1}\frac{\partial H_1}{\partial u_2}\frac{\partial H_3}{\partial u_1} + \frac{1}{H_2}\frac{\partial H_2}{\partial u_1}\frac{\partial H_3}{\partial u_2}, \end{aligned}\right. \tag{2}$$

and

$$\left\{\begin{aligned} \frac{\partial}{\partial u_1}\left(\frac{1}{H_1}\frac{\partial H_2}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{1}{H_2}\frac{\partial H_1}{\partial u_2}\right) + \frac{1}{H_3^2}\frac{\partial H_1}{\partial u_3}\frac{\partial H_2}{\partial u_3} &= 0, \\ \frac{\partial}{\partial u_2}\left(\frac{1}{H_2}\frac{\partial H_3}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{1}{H_3}\frac{\partial H_2}{\partial u_3}\right) + \frac{1}{H_1^2}\frac{\partial H_2}{\partial u_1}\frac{\partial H_3}{\partial u_1} &= 0, \\ \frac{\partial}{\partial u_3}\left(\frac{1}{H_3}\frac{\partial H_1}{\partial u_3}\right) + \frac{\partial}{\partial u_1}\left(\frac{1}{H_1}\frac{\partial H_3}{\partial u_1}\right) + \frac{1}{H_2^2}\frac{\partial H_3}{\partial u_2}\frac{\partial H_1}{\partial u_2} &= 0. \end{aligned}\right. \tag{3}$$

Moreover, it can be shown[†] that these necessary conditions are also sufficient so that when three functions are known which satisfy them there is a corresponding triply orthogonal system and it is unique.

In the future discussion it will be necessary to have the expressions for the principal radii of these surfaces, and also for the first and second curvatures of their curves of intersection. If we denote by ρ_{31} the principal radius of curvature of the surface $u_3 = \text{const.}$ along the curve $u_2 = \text{const.}$ and similarly for the others, we have[‡]

$$\begin{aligned} \frac{1}{\rho_{12}} &= \frac{1}{H_1 H_2}\frac{\partial H_2}{\partial u_1}, & \frac{1}{\rho_{23}} &= \frac{1}{H_2 H_3}\frac{\partial H_3}{\partial u_2}, & \frac{1}{\rho_{31}} &= \frac{1}{H_3 H_1}\frac{\partial H_1}{\partial u_3}, \\ \frac{1}{\rho_{13}} &= \frac{1}{H_1 H_3}\frac{\partial H_3}{\partial u_1}, & \frac{1}{\rho_{21}} &= \frac{1}{H_2 H_1}\frac{\partial H_1}{\partial u_2}, & \frac{1}{\rho_{32}} &= \frac{1}{H_3 H_2}\frac{\partial H_2}{\partial u_3}, \end{aligned} \tag{4}$$

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[*] Leçons sur les coordonnes curvilignes, p. 76; Bianchi, Lezioni, II, p. 479.

[†] Ib.

[‡] Bianchi, Lezioni, II, p. 482.

Again, if the first and second curvatures of the curves of parameter u_1 are denoted by $\frac{1}{R_1}$ and $\frac{1}{T_1}$, it is found that*

$$(5)\quad \begin{cases} \dfrac{1}{R_1^2}=\dfrac{1}{\rho_{21}^2}+\dfrac{1}{\rho_{31}^2}, & \dfrac{1}{R_2^2}=\dfrac{1}{\rho_{12}^2}+\dfrac{1}{\rho_{32}^2}, \quad \dfrac{1}{R_3^2}=\dfrac{1}{\rho_{23}^2}+\dfrac{1}{\rho_{13}^2}, \\ \dfrac{1}{T_1}=\dfrac{1}{H_1}\dfrac{\partial}{\partial u_1}\tan^{-1}\dfrac{\rho_{31}}{\rho_{21}}, & \dfrac{1}{T_2}=\dfrac{1}{H_2}\dfrac{\partial}{\partial u_2}\tan^{-1}\dfrac{\rho_{12}}{\rho_{32}}, \quad \dfrac{1}{T_3}=\dfrac{1}{H_3}\dfrac{\partial}{\partial u_3}\tan^{-1}\dfrac{\rho_{23}}{\rho_{13}}. \end{cases}$$

§2. *Systems with One Family of A-Surfaces.*

We have seen that for a surface S, whose linear element is written in the form

$$ds^2 = A^2du^2 + C^2dv^2$$

and the lines of curvature are parametric, to be an A-surface it is necessary and sufficient that the following equations be satisfied

$$(6)\qquad \frac{1}{A}\frac{\partial C}{\partial u}=\frac{\partial\omega}{\partial u}, \quad \frac{1}{C}\frac{\partial A}{\partial v}=-\frac{\partial\omega}{\partial v}, \quad \frac{\partial^2\omega}{\partial u^2}-\frac{\partial^2\omega}{\partial v^2}=\sin\omega\cos\omega.$$

In this case the linear element of the spherical representation has the form

$$d\sigma^2=\sin^2\omega\, du^2+\cos^2\omega\, dv^2.$$

For such a surface to belong to the surfaces $u_3=\text{const.}$ of a triple system ω must involve u_3 and from (6) it follows that A and C are determined only to within the same factor U_3, a function of u_3 alone. For each surface of the family this function would be a constant, and by replacing du and dv by $\frac{du}{U_3}$ and $\frac{dv}{U_3}$ respectively, this function would be made to disappear. Hence, the necessary and sufficient condition that the surfaces $u_3=\text{const.}$ be A-surfaces is that H_1 and H_2 satisfy the conditions

$$(7)\qquad -\frac{1}{H_2}\frac{\partial H_1}{\partial u_2}=\frac{\partial\omega}{\partial u_2}, \quad \frac{1}{H_1}\frac{\partial H_2}{\partial u_1}=\frac{\partial\omega}{\partial u_1},$$

where ω is a function of u_1, u_2, u_3, satisfying the equation

$$(8)\qquad \frac{\partial^2\omega}{\partial u_1^2}-\frac{\partial^2\omega}{\partial u_2^2}=\frac{\sin\omega\cos\omega}{U_3^2},$$

U_3 being an arbitrary function of u_3, and those equations of condition (2) and (3) which are independent of the above.

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* Bianchi, l. c., p. 484.

The spherical representation of the lines of curvature of the surfaces $u_3 = \text{const.}$ is

$$d\sigma^2 = \frac{\sin^2 \omega}{U_3^2} du_1^2 + \frac{\cos^2 \omega}{U_3^2} du_2^2. \tag{9}$$

From this and (1) we have the following expressions for the principal radii of the surfaces $u_3 = \text{const.}$

$$\frac{1}{\rho_{31}} = -\frac{\sin \omega}{U_3 H_1}, \quad \frac{1}{\rho_{32}} = \frac{\cos \omega}{U_3 H_2}. \tag{10}$$

When these expressions are compared with (4), we find that we must have

$$H_3 = -\frac{U_3}{\sin \omega} \frac{\partial H_1}{\partial u_3} = \frac{U_3}{\cos \omega} \frac{\partial H_2}{\partial u_3}, \tag{11}$$

so that the following condition is to be satisfied

$$\cos \omega \frac{\partial H_1}{\partial u_3} + \sin \omega \frac{\partial H_2}{\partial u_3} = 0. \tag{12}$$

In accordance with this we introduce a function ϕ defined by

$$\frac{\partial H_1}{\partial u_3} = -\phi \sin \omega \frac{\partial \omega}{\partial u_3}, \quad \frac{\partial H_2}{\partial u_3} = \phi \cos \omega \frac{\partial \omega}{\partial u_3}, \tag{13}$$

so that by quadratures and from (11) we have for the fundamental coefficients the following expressions

$$\begin{cases} H_1 = \phi \cos \omega - \int \cos \omega \frac{\partial \phi}{\partial u_3} du_3 + F_1(u_1, u_2), \\ H_2 = \phi \sin \omega - \int \sin \omega \frac{\partial \phi}{\partial u_3} du_3 + F_2(u_1, u_2), \\ H_3 = \phi U_3 \frac{\partial \omega}{\partial u_3}, \end{cases} \tag{14}$$

where the functions F_1 and F_2 do not involve u_3.

The first of equations (2) can be put in either of the forms

$$\frac{\partial}{\partial u_3}\left(\frac{1}{H_2} \frac{\partial H_1}{\partial u_2}\right) = \frac{1}{H_2 H_3} \frac{\partial H_3}{\partial u_2} \frac{\partial H_1}{\partial u_3},$$

$$\frac{\partial}{\partial u_2}\left(\frac{1}{H_3} \frac{\partial H_1}{\partial u_3}\right) = \frac{1}{H_2 H_3} \frac{\partial H_2}{\partial u_3} \frac{\partial H_1}{\partial u_2}.$$

If the preceding expressions be substituted in these equations, the former reduces to

$$\frac{1}{H_2}\frac{\partial H_3}{\partial u_2}=\frac{U_3}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3}, \tag{15}$$

and the latter vanishes identically. Again, the second of equations (2) takes either of the forms

$$\frac{\partial}{\partial u_3}\left(\frac{1}{H_1}\frac{\partial H_2}{\partial u_1}\right)=\frac{1}{H_1H_3}\frac{\partial H_3}{\partial u_1}\frac{\partial H_2}{\partial u_3},$$
$$\frac{\partial}{\partial u_1}\left(\frac{1}{H_3}\frac{\partial H_2}{\partial u_3}\right)=\frac{1}{H_1H_3}\frac{\partial H_1}{\partial u_3}\frac{\partial H_2}{\partial u_1},$$

of which the latter is satisfied identically in consequence of (11) and the former is reducible to

$$\frac{1}{H_1}\frac{\partial H_3}{\partial u_1}=\frac{U_3}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\partial u_3}. \tag{16}$$

The third of equations (2) can be written in either of the forms

$$\begin{aligned}\frac{\partial}{\partial u_2}\left(\frac{1}{H_1}\frac{\partial H_3}{\partial u_1}\right)&=\frac{1}{H_1H_2}\frac{\partial H_2}{\partial u_1}\frac{\partial H_3}{\partial u_2},\\ \frac{\partial}{\partial u_1}\left(\frac{1}{H_2}\frac{\partial H_3}{\partial u_2}\right)&=\frac{1}{H_1H_2}\frac{\partial H_1}{\partial u_2}\frac{\partial H_3}{\partial u_1},\end{aligned} \tag{17}$$

which by means of (15) and (16) can be given the forms

$$\left\{\begin{aligned}\frac{\partial}{\partial u_2}\left(\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\partial u_3}\right)&=\frac{1}{\sin\omega}\frac{\partial\omega}{\partial u_1}\frac{\partial^2\omega}{\partial u_2\partial u_3},\\ \frac{\partial}{\partial u_1}\left(\frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3}\right)&=-\frac{1}{\cos\omega}\frac{\partial\omega}{\partial u_2}\frac{\partial^2\omega}{\partial u_1\partial u_3}.\end{aligned}\right. \tag{18}$$

The first of equations (3) reduces to (8), and the second and third become in consequence of (7), (11), (15), and (16)

$$\left\{\begin{aligned}\frac{\partial}{\partial u_1}\left(\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\partial u_3}\right)&=\frac{1}{U_3}\frac{\partial}{\partial u_3}\left(\frac{\sin\omega}{U_3}\right)+\frac{1}{\sin\omega}\frac{\partial\omega}{\partial u_2}\frac{\partial^2\omega}{\partial u_2\partial u_3},\\ \frac{\partial}{\partial u_2}\left(\frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3}\right)&=-\frac{1}{U_3}\frac{\partial}{\partial u_3}\left(\frac{\cos\omega}{U_3}\right)-\frac{1}{\cos\alpha}\frac{\partial\omega}{\partial u_1}\frac{\partial^2\omega}{\partial u_1\partial u_3},\end{aligned}\right. \tag{19}$$

either of which is a consequence of the other and (8). We see then that ω must satisfy three equations, namely (8) and one each of (18) and (19).

From the preceding discussion it is seen that when ω satisfies these equations the only other conditions to be satisfied are equations (7) which become in consequence of (14)

$$(20)\quad \begin{aligned} &\frac{\partial\phi}{\partial u_2}\cos\omega + \int \sin\omega \frac{\partial\omega}{\partial u_2}\frac{\partial\phi}{\partial u_3}\,du_3 - \int \cos\omega \frac{\partial^2\phi}{\partial u_2 \partial u_3}\,du_3 + \frac{\partial F_1}{\partial u_2} \\ &\qquad\qquad + \frac{\partial\omega}{\partial u_2}\left[F_2 - \int \sin\omega \frac{\partial\phi}{\partial u_3}\,du_3\right] = 0, \\ &\frac{\partial\phi}{\partial u_1}\sin\omega - \int \cos\omega \frac{\partial\omega}{\partial u_1}\frac{\partial\phi}{\partial u_3}\,du_3 - \int \sin\omega \frac{\partial^2\phi}{\partial u_1 \partial u_3}\,du_3 + \frac{\partial F_2}{\partial u_1} \\ &\qquad\qquad - \frac{\partial\omega}{\partial u_1}\left[F_1 - \int \cos\omega \frac{\partial\phi}{\partial u_3}\,du_3\right] = 0. \end{aligned}$$

If these equations are differentiated with respect to u_3, they become

$$(21)\quad \begin{aligned} \sin\omega \frac{\partial\omega}{\partial u_3}\frac{\partial\phi}{\partial u_2} &= \left[F_2 - \int \sin\omega \frac{\partial\phi}{\partial u_3}\,du_3\right]\frac{\partial^2\omega}{\partial u_2 \partial u_3}, \\ \cos\omega \frac{\partial\omega}{\partial u_3}\frac{\partial\phi}{\partial u_1} &= \left[F_1 - \int \cos\omega \frac{\partial\phi}{\partial u_3}\,du_3\right]\frac{\partial^2\omega}{\partial u_1 \partial u_3}, \end{aligned}$$

which are merely equations (15) and (16) in another form.

Consider the particular case where ϕ is a constant which may be taken equal to unity. Equations (21) are satisfied if F_1 and F_2 are zero, and equations (20) vanish identically. The linear element (1) reduces then to

$$(22)\qquad ds^2 = \cos^2\omega\, du_1^2 + \sin^2\omega\, du_2^2 + U_3^2\left(\frac{\partial\omega}{\partial u_3}\right)^2 du_3^2,$$

from which it is seen that the surfaces $u_3 =$ constant are pseudospherical. Bianchi has shown* that, when ω satisfies equations (8), (18), and (19), the above linear element defines the most general triply orthogonal systems of which the surfaces $u_3 =$ const. are pseudospherical. Hence the theorem:

When one family of a triple system is composed of A-surfaces, the pseudospherical surfaces with the same spherical representation as the latter form a family of another triply orthogonal system.

Also,

Systems of lines on the unit sphere which are the images of the lines of curvature of pseudospherical surfaces forming a system of Lame, and these alone, are the spherical representation of the lines of curvature of A-surfaces forming such a system.

Such a triply-orthogonal system will be called an *A-system.*

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*Lezioni, II, p. 531.

When then there is such a family of lines, there is a great number of corresponding A-systems and their determination requires the solution of the equations (20).

As an example of the foregoing we inquire whether there is an A-system formed of parallels of pseudospherical surfaces. We have shown* that the parallels of such a surface are given by

$$H_1 = \cos\omega - a\sin\omega, \qquad H_2 = \sin\omega + a\cos\omega,$$

where a is a constant determining the various surfaces. For the case under consideration a may involve u_3. If the above expressions be substituted in equation (12), it reduces to

$$a\frac{\partial\omega}{\partial u_3} = 0.$$

From the expression (14) for H_3 it is seen that ω must involve u_3, so that the above equation is satisfied only when a is zero, that is, when the surfaces $u_3 =$ const. are pseudospherical.

§3. *Combescure-Darboux Transformations. Particular Solutions.*

From the foregoing discussion it follows that the determination of all the A-systems reduces to the finding of all pseudospherical systems and all systems related to the latter in such a way that the normals to the surfaces of the system are parallel to the normals to the corresponding surfaces of the pseudospherical system. Combescure and Darboux† have considered the problem of finding surfaces of Lame which have this correspondence with a given system, and we shall apply their results to our particular problem. We shall refer to it as the *Combescure-Darboux Transformation.*

With Darboux we put for the sake of brevity

$$\beta_{ij} = \frac{1}{H_i}\frac{\partial H_j}{\partial u_i}, \tag{23}$$

where i and j are different and assume the values 1, 2, 3. If we denote by X_1, Y_1, Z_1; $X_2 \dots$; X_3, Y_3, Z_3, the direction-cosines of the tangents to the curves of parameter u_1, u_2, u_3 respectively, it can be shown that‡

$$\begin{aligned} \frac{\partial X_k}{\partial u_i} &= \beta_{ki}X_i, \\ \frac{\partial X_k}{\partial u_k} &= -\beta_{ik}X_i - \beta_{lk}X_l, \end{aligned} \tag{23'}$$

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* Surfaces with the same, etc., l. c. p. 123.

† Sur les surfaces orthogonales, Annales de L'Ecole Normale, ser. 1, Vol. 3, p. 114.

‡ Lame, l. c., p. 74; Bianchi, Lezioni, II, p. 494.

where i, k, l are different and take the values 1, 2, 3. In terms of the functions β the equations of Lame (2) and (3) are expressed as follows:

$$\frac{\partial \beta_{ki}}{\partial u_i} = \beta_{ki}\beta_{li},$$

$$\frac{\partial \beta_{ik}}{\partial u_i} + \frac{\partial \beta_{ki}}{\partial u_k} + \beta_{il}\beta_{kl} = 0.$$

If a triple system is given, the corresponding functions β are given by (23) and the determination of the transformation requires the solution for H_1, H_2, H_3 of the six equations (23) after the functions β have been given the calculated values. When the system is pseudospherical, the forms of H_1, H_2, H_3 are as given by (22), so that the determination of the Combescure-Darboux transforms of a pseudospherical system requires the solution of the equations

$$(24) \qquad \begin{array}{lll} \dfrac{1}{H_1}\dfrac{\partial H_2}{\partial u_1} = \dfrac{\partial \omega}{\partial u_1}, & \dfrac{1}{H_2}\dfrac{\partial H_3}{\partial u_2} = \dfrac{U_3}{\sin\omega}\dfrac{\partial^2 \omega}{\partial u_2 \partial u_3}, & \dfrac{1}{H_3}\dfrac{\partial H_1}{\partial u_3} = -\dfrac{\sin\omega}{U_3}, \\ \dfrac{1}{H_2}\dfrac{\partial H_1}{\partial u_2} = -\dfrac{\partial \omega}{\partial u_2}, & \dfrac{1}{H_3}\dfrac{\partial H_2}{\partial u_3} = \dfrac{\cos\omega}{U_3}, & \dfrac{1}{H_1}\dfrac{\partial H_3}{\partial u_1} = \dfrac{U_3}{\cos\omega}\dfrac{\partial^2 \omega}{\partial u_1 \partial u_3}. \end{array}$$

These are the very equations of condition which we have found by the first method.

If we eliminate H_1 and H_2 from the above equations, we are brought to the following triple system to be satisfied by H_3:

$$(25) \qquad \left\{ \begin{array}{l} \dfrac{\partial^2 H_3}{\partial u_1 \partial u_2} = \dfrac{V}{W}\dfrac{\partial \omega}{\partial u_1}\dfrac{\partial H_3}{\partial u_1} - \dfrac{W}{V}\dfrac{\partial \omega}{\partial u_2}\dfrac{\partial H_3}{\partial u_2}, \\ \dfrac{\partial^2 H_3}{\partial u_1 \partial u_3} = \dfrac{\partial}{\partial u_3}\log(U_3 W)\dfrac{\partial H_3}{\partial u_1} - \sin\omega\, W H_3, \\ \dfrac{\partial^2 H_3}{\partial u_2 \partial u_3} = \dfrac{\partial}{\partial u_3}\log(U_3 V)\dfrac{\partial H_3}{\partial u_2} + \cos\omega\, V H_3, \end{array} \right.$$

where for the sake of brevity we have put

$$(26) \qquad V = \frac{1}{\sin\omega}\frac{\partial^2 \omega}{\partial u_2 \partial u_3}, \qquad W = \frac{1}{\cos\omega}\frac{\partial^2 \omega}{\partial u_1 \partial u_3}.$$

In consequence of the equations (8), (18), and (19) the conditions of integrability of the above equations are satisfied, and it is readily seen that the general solution of this system involves three arbitrary functions. The only question then as to the existence of A-systems comes back to the question as to the existence of a function ω satisfying equations (8), (18), and (19). This problem has been attacked by Bianchi in seeking to establish the existence of pseudospherical systems of Lame and he has not only proved that they exist but has given an idea of the degree of arbitrariness in the expression for ω.*

When a solution H_3 of equations (25) is known, the corresponding functions H_1 and H_2 are given directly by (24). When the values of X_i, Y_i, Z_i are known, the cartesian coordinates of space referred to the system corresponding to a given set of solutions of (24) are found by quadratures; namely

$$dx = \Sigma H_i X_i\, du_i, \qquad dy = \Sigma H_i Y_i\, du_i, \qquad dz = \Sigma H_i Z_i\, du_i.$$

If we substitute the expression for H_3 given in (14) in equations (25), they can be reduced to

$$(27)\quad \begin{cases} V\,.\,W\,.\,\dfrac{\partial\omega}{\partial u_3}\dfrac{\partial^2\phi}{\partial u_1\partial u_2} + V^2\left(\sin\omega\, W - \dfrac{\partial\omega}{\partial u_1}\dfrac{\partial\omega}{\partial u_3}\right)\dfrac{\partial\phi}{\partial u_1} \\ \qquad\qquad + W^2\left(\cos\omega\, V + \dfrac{\partial\omega}{\partial u_2}\dfrac{\partial\omega}{\partial u_3}\right)\dfrac{\partial\phi}{\partial u_2} = 0, \\ \dfrac{\partial^2\phi}{\partial u_1\partial u_3} + \dfrac{\partial}{\partial u_3}\log\dfrac{\frac{\partial\omega}{\partial u_3}}{W}\dfrac{\partial\phi}{\partial u_1} + \dfrac{\cos\omega\, W}{\frac{\partial\omega}{\partial u_3}}\dfrac{\partial\phi}{\partial u_3} = 0, \\ \dfrac{\partial^2\phi}{\partial u_2\partial u_3} + \dfrac{\partial}{\partial u_3}\log\dfrac{\frac{\partial\omega}{\partial u_3}}{V}\dfrac{\partial\phi}{\partial u_2} + \dfrac{\sin\omega\, V}{\frac{\partial\omega}{\partial u_3}}\dfrac{\partial\phi}{\partial u_3} = 0. \end{cases}$$

These equations can be derived readily from equations (20) and (21). We shall consider now several particular cases for which equations (27) take more simple forms.

It is seen that the above equations are satisfied when ϕ is a constant, and it was seen that if F_1 and F_2 are zero at the same time the system is pseudospherical. We inquire whether the latter is the only solution of this kind. For ϕ, a constant, equations (21) reduce to

$$F_1\frac{\partial^2\omega}{\partial u_1\,\partial u_3} = 0, \qquad F_2\frac{\partial^2\omega}{\partial u_2\,\partial u_3} = 0.$$

From (18) it follows that if $\dfrac{\partial^2\omega}{\partial u_2\partial u_3}$ is zero so also is $\dfrac{\partial^2\omega}{\partial u_1\partial u_3}$ or $\dfrac{\partial\omega}{\partial u_2}$. The former leads by means of (19) to

$$\frac{\partial}{\partial u_3}\frac{\cos\omega}{U_3} = \frac{\partial}{\partial u_3}\frac{\sin\omega}{U_3} = 0,$$

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*Lezioni, II, p. 532 et seq.

from which we have

$$\cos\omega = U_3 f_1(u_1, u_2), \qquad \sin\omega = U_3 f_2(u_2, u_3),$$

where the functions f_1 and f_2 do not involve u_3. This shows that U_3 is a constant and then ω would be independent of u_3, which we have seen to be impossible. The only other possibility then is that F_1 and $\frac{\partial\omega}{\partial u_2}$ are zero, in which case F_2 is an arbitrary function of u_2 alone, as is seen from (20). Hence the theorem:

The function ϕ *is constant for pseudospherical systems and for certain systems for which* F_1 *or* F_2 *is zero, and* F_2 *a function of* u_2 *alone, or* F_1 *a function of* u_1 *alone respectively; and these are the only cases.*

We inquire now whether ϕ can be a function of u_3 alone. From the second and third of equations (27) we see that in this case we must have

$$\frac{\partial^2\omega}{\partial u_1\,\partial u_3} = \frac{\partial^2\omega}{\partial u_2\,\partial u_3} = 0.$$

This has been seen to be impossible, hence *the function* ϕ *must involve one of the parameters* u_1, u_2 or *be a constant.*

This suggests finally the inquiry as to whether ϕ can be independent of u_2. From the third of equations (27) we have

$$\frac{\partial^2\omega}{\partial u_2\,\partial u_3} = 0,$$

in consequence of which the first of these equations is satisfied identically and ϕ must be an integral of the second. We have seen that the above condition is equivalent to the vanishing of $\frac{\partial\omega}{\partial u_2}$. The converse also is true, so that we have the theorem:

If one of the functions ϕ *and* ω *is independent of* u_1 *or* u_2, *so also is the other.*

§4. *Generalized Bäcklund Transformations of A-Systems.*

In our previous discussion[*] of A-surfaces we have established a transformation of these surfaces which reduces for the pseudospherical surfaces to the Bäcklund transformation. It may be defined as follows: Given an A-surface and a solution θ of the system

$$\sin\sigma\left(\frac{\partial\theta}{\partial u_1} + \frac{\partial\omega}{\partial u_2}\right) = \sin\theta\cos\omega - \cos\sigma\cos\theta\sin\omega,$$

$$\sin\sigma\left(\frac{\partial\theta}{\partial u_2} + \frac{\partial\omega}{\partial u_1}\right) = -\cos\theta\sin\omega + \cos\sigma\sin\theta\cos\omega,$$

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[*] Surfaces with the same etc., l. c. p. 148.

where σ is a constant. In the tangent plane to S draw the line through the point of contact which makes the angle θ with the tangent to the curve $u_2 =$ const. through the point. The plane which cuts the tangent plane along this line and under the angle σ envelopes another A-surface for which the spherical representation of the lines of curvature is

$$d\sigma_1^2 = \sin^2\theta \, du_1^2 + \cos^2\theta \, du_2^2.$$

We are now going to consider an A-system and determine the conditions to be satisfied in order that the transforms of the A-surfaces form part of a second triple system.

Just as equations (6) had to be given a more general form, namely (7) and (8), in the case of triple systems, so likewise the equations for the determination of θ take the form

$$(28) \qquad \begin{aligned} \varkappa\left(\frac{\partial\theta}{\partial u_1} + \frac{\partial\omega}{\partial u_2}\right) &= \sin\theta\cos\omega - \cos\sigma\cos\theta\sin\omega, \\ \varkappa\left(\frac{\partial\theta}{\partial u_2} + \frac{\partial\omega}{\partial u_1}\right) &= -\cos\theta\sin\omega + \cos\sigma\sin\theta\cos\omega, \end{aligned}$$

where we have put

$$\varkappa = U_3 \sin\sigma,$$

from which it follows that at most $\varkappa$ is a function of u_3. We denote by x_1, y_1, z_1 the coordinates of the point of tangency on the transform, by μ the distance from this point to the line of intersection of the two planes and by λ the distance from the foot of this perpendicular to the point of contact of the given surface S. From the expressions for λ and μ as found in the case of the transformation of a single surface* we have for the present case

$$\begin{aligned} \lambda &= \varkappa\,(H_1\cos\omega + H_2\sin\omega), \\ \mu &= \varkappa\,(-H_1\sin\omega + H_2\cos\omega). \end{aligned}$$

Hence we have for the coordinates of the transform

$$x_1 = x + (\lambda\cos\theta - \mu\cos\sigma\sin\theta)\,X_1 + (\lambda\sin\theta + \mu\cos\sigma\cos\theta)\,X_2 - \mu\sin\sigma X_3$$

and analogous expressions for y_1 and z_1, X_1, X_2, X_3 having the interpretation previously defined. From (23′) and the expressions for the functions β given by (23) and (24) we find readily

$$\begin{aligned}
&\frac{\partial X_1}{\partial u_1} = \frac{\partial\omega}{\partial u_2}X_2 + \frac{\sin\omega}{U_3}X_3, &&\frac{\partial X_1}{\partial u_2} = \frac{\partial\omega}{\partial u_1}X_2, &&\frac{\partial X_1}{\partial u_3} = \frac{1}{H_1}\frac{\partial H_3}{\partial u_1}X_3; \\
&\frac{\partial X_2}{\partial u_1} = -\frac{\partial\omega}{\partial u_2}X_1, &&\frac{\partial X_2}{\partial u_2} = -\frac{\partial\omega}{\partial u_1}X_1 - \frac{\cos\omega}{U_3}X_3, &&\frac{\partial X_2}{\partial u_3} = \frac{1}{H_2}\frac{\partial H_3}{\partial u_2}X_3; \\
&\frac{\partial X_3}{\partial u_1} = -\frac{\sin\omega}{U_3}X_1, &&\frac{\partial X_3}{\partial u_2} = \frac{\cos\omega}{U_3}X_2, &&\frac{\partial X_3}{\partial u_3} = -\frac{1}{H_1}\frac{\partial H_3}{\partial u_1}X_1 - \frac{1}{H_2}\frac{\partial H_3}{\partial u_2}X_2.
\end{aligned}$$

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*l. c. p. 150.

If we make use of these relations in our calculation of the differential quotients of x_1 and for the sake of brevity put

$$p = \frac{\partial H_1}{\partial u_1} + H_2 \frac{\partial \omega}{\partial u_1}, \qquad q = \frac{\partial H_2}{d u_2} - H_1 \frac{\partial \omega}{\partial u_2},$$

we find

$$(29)\quad \left\{ \begin{aligned} \frac{\partial x_1}{\partial u_1} &= \frac{1}{\varkappa} \left[\varkappa^2 p + \lambda \cos\theta - \mu \cos\sigma \sin\theta \right] \Big[(\cos\theta \cos\omega + \cos\sigma \sin\theta \sin\omega) X_1 \\ &\qquad + (\sin\theta \cos\omega - \cos\sigma \cos\theta \sin\omega) X_2 + \sin\sigma \sin\omega \, X_3 \Big], \\ \frac{\partial x_1}{\partial u_2} &= \frac{1}{\varkappa} \left[\varkappa^2 q + \lambda \sin\theta + \mu \cos\sigma \cos\theta \right] \Big[(\cos\theta \sin\omega - \cos\sigma \sin\theta \cos\omega) X_1 \\ &\qquad + (\sin\theta \sin\omega + \cos\sigma \cos\theta \cos\omega) X_2 - \sin\sigma \cos\omega \, X_3 \Big], \\ \frac{\partial x_1}{\partial u_3} &= \Big[(\lambda \sin\theta \cos\sigma + \mu \cos\theta) \frac{\partial \omega}{\partial u_3} - \varkappa \cos\sigma \sin\theta \frac{H_3}{U_3} \\ &\quad - (\lambda \sin\theta + \mu \cos\sigma \cos\theta) \frac{\partial \theta}{\partial u_3} + (\lambda \cos\theta - \mu \cos\sigma \sin\theta) \frac{\partial \log \varkappa}{\partial u_3} \\ &\quad + \frac{\mu \sin\sigma}{H_1} \frac{\partial H_3}{\partial u_1} \Big] X_1 + \Big[- (\lambda \cos\theta \cos\sigma - \mu \sin\theta) \frac{\partial \omega}{\partial u_3} + \varkappa \cos\sigma \cos\theta \frac{H_3}{U_3} \\ &\quad + (\lambda \cos\theta - \mu \cos\sigma \sin\theta) \frac{\partial \theta}{\partial u_3} + (\lambda \sin\theta + \mu \cos\sigma \cos\theta) \frac{\partial \log \varkappa}{\partial u_3} \\ &\quad + \frac{\mu \sin\sigma}{H_2} \frac{\partial H_3}{\partial u_2} \Big] X_2 + \Big[H_3 \cos^2\sigma + (\lambda \cos\theta - \mu \cos\sigma \sin\theta) \frac{1}{H_1} \frac{\partial H_2}{\partial u_1} \\ &\quad + (\lambda \sin\theta + \mu \cos\sigma \cos\theta) \frac{1}{H_2} \frac{\partial H_1}{\partial u_2} + \lambda \sin\sigma \frac{\partial \omega}{\partial u_3} - \mu \sin\sigma \frac{\partial \log \varkappa}{\partial u_3} \Big] X_3 . \end{aligned} \right.$$

From these expressions it is readily found that

$$\Sigma \frac{\partial x_1}{\partial u_1} \frac{\partial x_1}{\partial u_2} = 0,$$

and by direct calculation we find that

$$\begin{aligned} \sum \frac{\partial x_1}{\partial u_1} \frac{\partial x_1}{\partial u_3} &= H_2 (p \varkappa^2 + \lambda \cos\theta - \mu \cos\sigma \cos\theta) \Big[\frac{\partial \omega}{\partial u_3} - \cos\sigma \frac{\partial \theta}{\partial u_3} + \frac{\sin\sigma \cos\theta}{H_1} \frac{\partial H_3}{\partial u_1} \\ &\qquad + \frac{\sin\sigma \sin\theta}{H_2} \frac{\partial H_3}{\partial u_2} - \frac{H_2}{H_1} \frac{\partial \log \varkappa}{\partial u_3} \Big], \\ \sum \frac{\partial x_1}{\partial u_2} \frac{\partial x_1}{\partial u_3} &= - H_1 (q \varkappa^2 + \lambda \sin\theta + \mu \cos\sigma \sin\theta) \Big[\frac{\partial \omega}{\partial u_3} - \cos\sigma \frac{\partial \theta}{\partial u_3} + \frac{\sin\sigma \cos\theta}{H_1} \frac{\partial H_3}{\partial u_1} \\ &\qquad + \frac{\sin\sigma \sin\theta}{H_2} \frac{\partial H_3}{\partial u_2} + \frac{H_1}{H_2} \frac{\partial \log \varkappa}{\partial u_3} \Big]. \end{aligned}$$

From these expressions and the above for $\frac{\partial x_1}{\partial u_1}$ and $\frac{\partial x_1}{\partial u_2}$ it follows that for the transformed system to be triply orthogonal it is necessary that the expressions in the brackets vanish. These equations lead to

$$\frac{\partial \log \varkappa}{\partial u_3} = 0,$$

from which it follows that $\varkappa$ is a constant, and to

$$\cos \sigma \frac{\partial \theta}{\partial u_3} = \frac{\sin \sigma \cos \theta}{H_1} \frac{\partial H_3}{\partial u_1} + \frac{\sin \sigma \sin \theta}{H_2} \frac{\partial H_3}{\partial u_2} + \frac{\partial \omega}{\partial u_3}$$

which must be satisfied by θ in addition to (28). By means of equations (15) and (16) this equation can be given the form

$$(30) \qquad \cos \sigma \frac{\partial \theta}{\partial u_3} = \frac{\varkappa \cos \theta}{\cos \omega} \frac{\partial^2 \omega}{\partial u_1 \partial u_3} + \frac{\varkappa \sin \theta}{\sin \omega} \frac{\partial^2 \omega}{\partial u_2 \partial u_3} + \frac{\partial \omega}{\partial u_3}.$$

It is important to note that equations (28) and (30) involve only ω, σ, $\varkappa$, and hence θ does not vary with the Combescure-Darboux transformation. We have then the theorem:

Whenever a function θ is known which transforms a given A-system into a similar system, the same function determines a like transformation for all the Combescure-Darboux transforms of the given system.

In consequence of (30) the expression for $\frac{\partial x_1}{\partial u_3}$ can be reduced to

$$\frac{\partial x_1}{\partial u_3} = \Big[\lambda \sin \sigma \frac{\partial \theta}{\partial u_3} + \cos \sigma\, H_3 + (\lambda \cos \theta \cos \sigma - \mu \sin \theta) \frac{U_3}{\cos \omega} \frac{\partial^2 \omega}{\partial u_1 \partial u_3} + (\lambda \sin \theta \cos \sigma$$
$$+ \mu \cos \theta) \frac{U_3}{\sin \omega} \frac{\partial^2 \omega}{\partial u_2 \partial u_3}\Big] (-\sin \sigma \sin \theta\, X_1 + \sin \sigma \cos \theta\, X_2 + \cos \sigma\, X_3).$$

For the linear element of space referred to the transformed system we find from the above

$$ds_1^2 = \frac{1}{\varkappa^2} (\varkappa^2 p + \lambda \cos \theta - \mu \cos \sigma \sin \theta)^2 du_1^2 + \frac{1}{\varkappa^2} (\varkappa^2 q + \lambda \sin \theta + \mu \cos \sigma \cos \theta)^2 du_2^2$$
$$+ \Big[\lambda \sin \sigma \frac{\partial \theta}{\partial u_3} + \cos \sigma\, H_3 + (\lambda \cos \theta \cos \sigma - \mu \sin \theta) \frac{U_3}{\cos \omega} \frac{\partial^2 \omega}{\partial u_1 \partial u_3}$$
$$+ (\lambda \sin \theta \cos \sigma + \mu \cos \theta) \frac{U_3}{\sin \omega} \frac{\partial^2 \omega}{\partial u_2 \partial u_3}\Big]^2 du_3^2,$$

which gives for the surfaces $u_3 = \text{const.}$ the expression for the linear element which we have found before.

The system (28) and (30), which must be satisfied by θ, can be replaced by the total differential equation

$$
\begin{aligned}
d\theta = & \left(\frac{\sin\theta\cos\omega - \cos\sigma\cos\theta\sin\omega}{\varkappa} - \frac{\partial\omega}{\partial u_2}\right) du_1 \\
(31) \qquad & - \left(\frac{\cos\theta\sin\omega - \cos\sigma\sin\theta\cos\omega}{\varkappa} + \frac{\partial\omega}{\partial u_1}\right) du_2 \\
& + \frac{\varkappa}{\cos\sigma}\left(\frac{\cos\theta}{\cos\omega}\frac{\partial^2\omega}{\partial u_1 \partial u_3} + \frac{\sin\theta}{\sin\omega}\frac{\partial^2\omega}{\partial u_2 \partial u_3} + \frac{1}{\varkappa}\frac{\partial\omega}{\partial u_3}\right).
\end{aligned}
$$

Since ω satisfies equations (8), (18), and (19), the conditions of integrability of this equation are satisfied, so that for each value of $\varkappa$ there is a general integral involving an arbitrary constant; hence corresponding to each value of $\varkappa$ there is an infinity of transformations of a given A-system into similar systems.

Bianchi[*] has called attention to the fact that if we introduce a new dependent variable in (31), defined by

$$\tan\frac{\theta}{2} = \Lambda,$$

the equation in Λ is of the Riccati form and consequently when a particular integral is known the equation can be integrated by quadratures. From this it follows that if a transform of a given system is known, all the transforms can be found by quadratures. But the latter admit the given system for a transform, and hence when the transforms of a given systen are known the transforms of the former are given by quadratures. We shall find furthermore that even these quadratures are unnecessary.

We have shown that if a surface of Bonnet S is transformed by means of θ_1 and σ_1 into S_1 and by means of θ_2 and σ_2 into S_2, there is a function ϕ given by

$$\tan\frac{\phi - \omega}{2} = \frac{\sin\dfrac{\sigma_1 + \sigma_2}{2}}{\sin\dfrac{\sigma_1 - \sigma_2}{2}} \tan\frac{\theta_1 - \theta_2}{2},$$

by means of which and σ_2 the surface S_1 can be transformed into a surface S_3, and S_2 can be transformed into the same surface S_3 by means of ϕ and σ_1. Consider now an A-system Σ and the associated pseudospherical system P. Let θ_1 be a particular solution of equation (31) for $\sigma = \sigma_1$ and θ_2 a particular solution for $\sigma = \sigma_2$. The former enables us to transform Σ into another A-system Σ_1 and

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[*] Lezioni, II, p. 540.

P into the associated pseudospherical system P_1; in like manner we pass from Σ by means of θ_2 to an A-system Σ_2 and from P to the associated pseudospherical system P_2. If the surfaces of Σ_1 are transformed by means of ϕ and σ_2 we get a family Σ_3 which is the same as is obtained from Σ_2 by means of ϕ and σ_1. In a similar manner we get from P_1 and P_2 the same pseudospherical system P_3. But Bianchi has shown* that the system P_3 is a family of Lame, and we have shown that the function which transforms one family of Lame of A-surfaces into a family of this type leads to a similar transformation of all the Combescure-Darboux transforms of the given family. Hence we have the following theorem of permutability for A-systems:

If Σ_1 and Σ_2 are two triply orthogonal A-systems bound to a system Σ by two transformations of Bäcklund of constants κ_1 and κ_2, there is a transformed system of B_1 of constant κ_2 which coincides with a transformed system of B_2 of constant κ_1.

From this result we pass at once to the following theorem:

If one knows how to find all the transforms of an A-system for all values of κ, the determination of the transforms of the latter requires only algebraic processes and differentiation.

§5. *Weingarten Systems of A-Surfaces.*

When in particular the function U_3 is a constant, which without loss of generality can be taken as unity,† the equations (8), (18), and (19) reduce to

$$(32)\quad \begin{cases} \dfrac{\partial^2\omega}{\partial u_1^2}-\dfrac{\partial^2\omega}{\partial u_2^2}=\sin\omega\cos\omega, \\[2ex] \dfrac{\partial}{\partial u_1}\left(\dfrac{1}{\cos\omega}\,\dfrac{\partial^2\omega}{\partial u_1\,\partial u_3}\right)=\cos\omega\,\dfrac{\partial\omega}{\partial u_3}+\dfrac{1}{\sin\omega}\,\dfrac{\partial\omega}{\partial u_2}\,\dfrac{\partial^2\omega}{\partial u_2\partial u_3}, \\[2ex] \dfrac{\partial}{\partial u_2}\left(\dfrac{1}{\sin\omega}\,\dfrac{\partial^2\omega}{\partial u_2\,\partial u_3}\right)=\sin\omega\,\dfrac{\partial\omega}{\partial u_3}-\dfrac{1}{\cos\omega}\,\dfrac{\partial\omega}{\partial u_1}\,\dfrac{\partial^2\omega}{\partial u_1\partial u_3}, \\[2ex] \dfrac{\partial^3\omega}{\partial u_1\partial u_2\partial u_3}=\cot\omega\,\dfrac{\partial\omega}{\partial u_1}\,\dfrac{\partial^2\omega}{\partial u_2\partial u_3}-\tan\omega\,\dfrac{\partial\omega}{\partial u_2}\,\dfrac{\partial^2\omega}{\partial u_1\partial u_3}. \end{cases}$$

Bianchi‡ has shown that this system may be replaced by the equivalent system

$$(33)\quad \begin{cases} \dfrac{\partial^2\omega}{\partial u_1^2}-\dfrac{\partial^2\omega}{\partial u_2^2}=\sin\omega\cos\omega, \\[2ex] \left(\dfrac{1}{\cos\omega}\,\dfrac{\partial^2\omega}{\partial u_1\,\partial u_3}\right)^2+\left(\dfrac{1}{\sin\omega}\,\dfrac{\partial^2\omega}{\partial u_2\partial u_3}\right)^2-\left(\dfrac{\partial\omega}{\partial u_3}\right)^2=F(u_3), \end{cases}$$

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* Lezioni, II, p. 541.

† Bianchi shows, l. c., p. 528, that the total curvature of one of the surfaces $u_3=$ const. is $-1/R^2$.

‡ Lezioni, II, p. 550.

where $F(u_3)$ is a function of u_3 alone. Three cases present themselves according as this function is positive, zero, or negative. As no restriction has been placed thus far upon the parameter u_3, it follows that it can be so chosen that we have the three cases

$$F(u_3) = +1, \qquad F(u_3) = 0, \qquad F(u_3) = -1. \tag{34}$$

From (4) and (22) it follows that, for all of these cases, when the surfaces $u_3 =$ const. are pseudospherical we have

$$\frac{1}{\rho_{31}} = -\tan\omega, \qquad \frac{1}{\rho_{32}} = \cot\omega$$

and consequently all the surfaces $u_3 =$ const. have the same total curvature. Since Weingarten was the first to discover the existence of pseudospherical systems with this property, Bianchi has called them the *systems of Weingarten.* In consequence of this we shall refer to all systems for which U_3 is constant as *Weingarten A-systems.*

Bianchi* has shown that a Bäcklund transformation of a system of Weingarten of any one of the three types (34) leads to a system of Weingarten of the same type. Since all of this refers only to the spherical representation we have the theorem :

Bäcklund transformations of an A-system of any of the Weingarten types give systems of the same kind.

If we denote by θ the angle which the principal normal to a curve of parameter u_3, drawn in the positive direction, at the point where it meets S makes with the positive direction of the tangent to the curve $u_2 =$ const. on S through the point, it is found that

$$\cos\theta = -\frac{R_3}{H_1}\frac{\partial}{\partial u_1}\log H_3,$$
$$\sin\theta = -\frac{R_3}{H_2}\frac{\partial}{\partial u_2}\log H_3,$$

which, in consequence of (15) and (16), become for the case where $F(u_3)$ is zero in (33)

$$\cos\theta\frac{\partial\omega}{\partial u_3} = -\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\,\partial u_3}, \qquad \sin\theta\frac{\partial\omega}{\partial u_3} = -\frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\,\partial u_3}.$$

Since ω must satisfy equations (18), we get that θ must satisfy the system

$$\tag{34'} \begin{aligned} \frac{\partial\omega}{\partial u_1} + \frac{\partial\theta}{\partial u_2} &= -\cos\theta\sin\omega, \\ \frac{\partial\omega}{\partial u_2} + \frac{\partial\theta}{\partial u_1} &= \sin\theta\cos\omega, \end{aligned}$$

---

† Ib., p. 562.

and from the preceding we have

$$\frac{\cos\theta}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\,\partial u_3}+\frac{\sin\theta}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\,\partial u_3}+\frac{\partial\omega}{\partial u_3}=0.$$

But these are the forms which equations (28) and (30) take for $\varkappa$ equal to unity and σ a right angle. Again, since $F(u_3)$ is zero, the second of equations (33) is of such a form that there is only one value of θ satisfying the above equations. Hence the theorem:

The osculating planes of the curves of parameter u_3, at the intersections of the latter with the surfaces $u_3 = const.$, envelope complementary transforms of the surfaces $u_3 = const.$ and form the only system of Lame composed of such transforms of the given system.

Since it was seen that $\varkappa$ in (28) is a constant, it follows that for σ to be a right angle U_3 must be constant. We have just seen that when F is zero there is only one transformation of this kind. In a similar manner it can be shown that when F is positive there are two transformations, and none when F is negative.

§6. *Systems of Moulure Surfaces. Lines of Equidistance.*

We have seen that, when ω does not involve u_2, ϕ also is independent of this parameter. We shall now consider this case.

Equations (8), (18), and (19) which determine ω reduce to

$$\frac{\partial^2\omega}{\partial u_1^2}-\frac{\partial^2\omega}{\partial u_2^2}=\frac{\sin\omega\cos\omega}{U_3^2},$$

$$\frac{1}{U_3}\frac{\partial}{\partial u_3}\left(\frac{\cos\omega}{U_3}\right)+\frac{1}{\cos\omega}\frac{\partial\omega}{\partial u_1}\frac{\partial^2\omega}{\partial u_1\,\partial u_3}=0,$$

$$\frac{\partial}{\partial u_1}\left(\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\,\partial u_3}\right)=\frac{1}{U_3}\frac{\partial}{\partial u_3}\left(\frac{\sin\omega}{U_3}\right),$$

of which the last is a direct consequence of the first two and these can be replaced by

$$\left(\frac{\partial\omega}{\partial u_1}\right)^2+\frac{\cos^2\omega}{U_3^2}=c, \tag{35}$$

where c is a constant.

The general integral of this equation is

$$\cos\omega=\mathrm{sn}\,(\tau,\varkappa)$$

where

$$\tau=\frac{u_1}{a}+f(u_3),\qquad \varkappa=\frac{a}{U_3},$$

the functions f and U_3 being arbitrary functions of u_3 and a an arbitrary constant. From (7) and (15) it follows that H_1 and H_3 are independent of u_2, and hence from (14) it is seen that F_1 is a function of u_1 alone. The first and last of equations (27) for the determination of ϕ are satisfied identically and the second reduces to

$$(36)\qquad \frac{\dfrac{\partial\omega}{\partial u_3}}{W}\,\frac{\partial^2\phi}{\partial u_1\,\partial u_3}+\frac{\partial}{\partial u_3}\left(\frac{\dfrac{\partial\omega}{\partial u_3}}{W}\right)\frac{\partial\phi}{\partial u_1}+\cos\omega\,\frac{\partial\phi}{\partial u_3}=0.$$

For each integral of this equation the corresponding functions F_1 and F_2 are given by (20) and (21) and then the triple system is completely determined.

Since ω does not involve u_2, the spherical images of the lines of curvature $u_2=$ const. on the surfaces $u_3=$ const. are great circles with a common diameter and consequently the latter are *moulure* surfaces, and for the surfaces the curves $u_2=$ const. are geodesics, which is evident analytically owing to the fact that H_1 does not involve u_2. Moreover, the functions H_3 are independent of u_2 and consequently the lines of curvature $u_2=$ const. on the surfaces $u_1=$ const. are geodesics. Again, from the general expressions (4) for the principal radii of the surfaces $u_2=$ const. it is seen that the latter surfaces are planes—the planes of the lines of curvature in one system on the surfaces $u_3=$ const. and $u_1=$ const. From this it follows that the latter surfaces also are moulure surfaces with the same cylindrical generator as the surfaces $u_3=$ const. Since the most general moulure surface is generated by a plane curve whose plane rolls without sliding over a cylinder, it is evident that the most general triple system with a family of moulure surfaces is obtained by tracing a family of curves and their orthogonal trajectories in a plane and rolling the plane over the cylinder, thus generating two families of moulure surfaces which together with the different positions of the plane form a triply orthogonal system.

Consider now the case of Weingarten systems of moulure surfaces; without loss of generality we can put U_3 and c equal to unity in (35), in which case the latter reduces to

$$(37)\qquad \frac{\partial\omega}{\partial u_1}=\sin\omega,$$

which can readily be integrated giving

$$(38)\qquad \tan\frac{\omega}{2}=e^{u_1+f(u_3)},$$

where f is an arbitrary function of u_3; but as no particular choice has been made of the parameter u_3 in deriving the general equations (8), (18), and (19), it follows that in all generality this function of u_3 can be replaced by u_3 alone. When this value for ω is substituted in equation (36), the latter becomes

$$\frac{\partial}{\partial u_1}\left(\sin\omega\frac{\partial\phi}{\partial u_3}\right)=0,$$

of which the integral is

$$\phi=\int\frac{\chi(u_3)}{\sin\omega}\,du_3+\psi(u_1),$$

where χ and ψ are arbitrary functions. When this substitution is made in the second of equations (21), it is found that

$$F_1=\frac{\partial\psi}{\partial u_1}$$

and from the first of equations (20) it follows that F_2 is an arbitrary function of u_2. Gathering together these results we have the following expressions for the fundamental coefficients of a general Weingarten system of moulure surfaces:

$$(39)\qquad\left\{\begin{aligned}H_1&=\cos\omega\left[\int\frac{\chi\,du_3}{\sin\omega}+\psi\right]-\int\frac{\chi\cos\omega}{\sin\omega}du_3+\psi',\\H_2&=\sin\omega\left[\int\frac{\chi\,du_3}{\sin\omega}+\psi\right]-\int\chi\,du_3+F_2(u_2),\\H_3&=\left[\int\frac{\chi\,du_3}{\sin\omega}+\psi\right]\frac{\partial\omega}{\partial u_3},\end{aligned}\right.$$

where ψ, F_2, and χ are arbitrary functions of u_1, u_2, and u_3 respectively.

We have shown elsewhere* that the linear element of a moulure surface can be given the form

$$ds^2=U_1^2du_1^2+\left[U_2-\int U_1\cos u_1\,du_1\right]du_2^2,$$

where U_1 and U_2 are functions of u_1 and u_2 respectively and the function $\cos u_1$ arises from the choice of the spherical representation. At the same time it was found that the function U_2 determines the character of the cylindrical generator and U_1 the form of the generating curve of the surface. Comparing this result with (39), we note that the functions χ and ψ determine the families of curves generating the two families of moulure surfaces and F_2 fixes the form of the cylinder.

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*Surfaces whose lines of curvature in one system are represented on the sphere by great circles. Amer. Journ. Vol. 25, p. 359.

In consequence of equations (15) and (16), the second of equations (33) can be put in the form

$$\left(\frac{1}{H_1}\frac{\partial H_3}{\partial u_1}\right)^2+\left(\frac{1}{H_2}\frac{\partial H_3}{\partial u_2}\right)^2-\frac{H_3^2}{\phi_2}=F(u_3),$$

or

$$\Delta H_3=F(u_3)+\frac{H_3^2}{\phi^2}, \tag{40}$$

where ΔH_3 denotes the first Beltrami differential parameter formed with respect to the linear element of the surfaces $u_3=\text{const.}$

$$ds^2=H_1^2\,du_1^2+H_2\,du_2^2.$$

Upon these surfaces there is a family of curves, defined by the equation $H_3=\text{const.}$, which have the property, as is seen from (1), that along such a curve the distances to the next surface of the system are constant; on this account they are called the curves of equidistance. From the theory of geodesic lines* and equation (40), it follows that the curves of equidistance on the surfaces $u_3=\text{const.}$ are geodesic parallels when ϕ is a constant or a function of H_3, or what is the same thing as the latter when ϕ is a function of $\frac{\partial\omega}{\partial u_3}$. Consider now the case

$$\phi=f\left(\frac{\partial\omega}{\partial u_3}\right),$$

whereupon equations (21) become

$$\begin{aligned}\sin\omega\,\frac{\partial\omega}{\partial u_3}\,\frac{\partial^2\omega}{\partial u_2\partial u_3}f'&=\left(F_2-\int\sin\omega\,\frac{\partial\phi}{\partial u_3}\,du_3\right)\frac{\partial^2\omega}{\partial u_2\partial u_3},\\ \cos\omega\,\frac{\partial\omega}{\partial u_3}\,\frac{\partial^2\omega}{\partial u_1\partial u_3}f'&=\left(F_1-\int\cos\omega\,\frac{\partial\phi}{\partial u_3}\,du_3\right)\frac{\partial^2\omega}{\partial u_1\partial u_3}.\end{aligned}\tag{41}$$

If neither $\frac{\partial^2\omega}{\partial u_1\partial u_3}$ nor $\frac{\partial^2\omega}{\partial u_2\partial u_3}$ is zero, these equations reduce to forms from which f' can be eliminated giving

$$\tan\omega\left(F_1-\int\cos\omega\,\frac{\partial\phi}{\partial u_3}\,du_3\right)=F_2-\int\sin\omega\,\frac{\partial\phi}{\partial u_3}\,du_3.$$

Differentiate with respect to u_3; this gives

$$\sec^2\omega\,\frac{\partial\omega}{\partial u_3}\left(F_1-\int\cos\omega\,\frac{\partial\phi}{\partial u_3}\,du_3\right)=0,$$

which is possible only in case ϕ is constant and F_1 is zero, which leads by the former equation to the necessity of the vanishing of F_2. We have seen that these conditions characterize the pseudospherical systems of Weingarten. We have seen that both $\frac{\partial^2\omega}{\partial u_1 \partial u_3}$ and $\frac{\partial^2\omega}{\partial u_2 \partial u_3}$ cannot be zero, so it only remains to consider the case where the latter vanishes; then $\frac{\partial\omega}{\partial u_2}$ vanishes, and consequently the surfaces $u_3 =$ const. are moulure surfaces. The second of equations (41) becomes

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*Bianchi, Lezioni, I, p. 193.

$$\cos\omega \frac{\partial\omega}{\partial u_3} f' = F_1 - \int \cos\omega \frac{\partial\phi}{\partial u_3} du_3.$$

Differentiate with respect to u_3; the result may be written

$$\frac{\partial^2\omega}{\partial u_3^2}\left(\frac{2f' + f''\frac{\partial\omega}{\partial u_3}}{f'\frac{\partial\omega}{\partial u_3}}\right) = \tan\omega \frac{\partial\omega}{\partial u_3},$$

where primes denote differentiation with respect to the argument. This equation can be integrated with respect to u_3 with the result

$$\Psi\left(\frac{\partial\omega}{\partial u_3}\right) = -\log\lambda\cos\omega,$$

where the form of the function Ψ is not essential and λ is a function independent of u_3; this equation may be written

$$\frac{\partial\omega}{\partial u_3} = F(\lambda\cos\omega). \tag{42}$$

Since ω is independent of u_2, λ also is independent of this parameter. Expressing the condition of integrability of equations (37) and (42), we get

$$\cos\omega\, F = \left(\frac{\partial\lambda}{\partial u_1}\cos\omega - \sin^2\omega\,\lambda\right)F'.$$

As F' cannot be zero, we write this equation in the form

$$\frac{F}{F'} = \frac{\partial\lambda}{\partial u_1} - \frac{\sin^2\omega}{\cos\omega}\lambda, \tag{43}$$

and differentiate it with respect to u_3; this gives

$$\frac{FF''}{F'^2} = -\sec^2\omega. \tag{44}$$

Now differentiate (43) with respect to u_1 and in the reduction make use of (44); this gives

$$2\frac{\partial\lambda}{\partial u_1} = \cos\omega\frac{\partial^2\lambda}{\partial u_1^2}.$$

Since ω must involve u_3, this equation can be satisfied only when λ is constant; hence equation (42) can be written

$$\frac{\partial\omega}{\partial u_3}=F(\cos\omega),$$

in which case equation (43) reduces to

$$\frac{F'}{F}=-\frac{\cos\omega}{\sin^2\omega},$$

from which it follows that in the most general case F is equal to *sin* ω. This result and (37) gives for ω the value

$$\tan\frac{\omega}{2}=e^{u_1+u_3},$$

which has been seen to be the value for all Weingarten systems of moulure surfaces. Hence the theorem:

The lines of equidistance in a Weingarten system are geodesic parallels when the surfaces $u_3 =$ const. are pseudospherical or moulure, and only for these systems.

§7. *A-Systems with the Curves of Parameter u_3 Plane.*

It has been seen that when the surfaces $u_3 =$ const. of an A-system are moulure surfaces the curves of parameter u_3 are plane curves. We seek now for the other A-systems having this property and shall exclude from the discussion the former systems, that is, we assume that ω involves all three parameters.

From the expression (5) for the torsion of the curves of parameter u_3 and the fundamental equations (15) and (16) it follows that the necessary and sufficient condition that these curves be plane is that ω satisfy the equation

$$\frac{\partial}{\partial u_3}\left(\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\partial u_3}\bigg/\frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3}\right)=0. \tag{45}$$

This equation can be replaced by the system

$$\left\{\begin{aligned}\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\partial u_3}&=\lambda\cos\alpha\frac{\partial\omega}{\partial u_3},\\ \frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3}&=\lambda\sin\alpha\frac{\partial\omega}{\partial u_3},\end{aligned}\right. \tag{46}$$

where α is a determined function of u_1 and u_2, and λ is an auxiliary function.

Ribaucour* was the first to show that if one considers a surface S of one of the families of a triple system and constructs the osculating circles of the orthogonal trajectories of S at the points of intersection with the latter, the double infinity of circles form a cyclic system. Moreover, Bianchi* shows that, if the orthogonal trajectories of S are plane curves, then for the triply orthogonal system of which the surfaces orthogonal to the cyclic system constitute one family the lines of curvature have the same spherical representation as for the given triple system. From this it follows that if the condition (45) is satisfied there is a corresponding A-system for which the curves of parameter u_3 are circles. We proceed to the consideration of this special system.

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*Bianchi, Lezioni, II, p. 487.

From (5), (14), (15), and (16) we have for the radius of principal curvature of the curves of parameter u_3 the expression

$$\frac{1}{R_3^2}=\frac{1}{\phi^2\left(\frac{\partial\omega}{\partial u_3}\right)^2}\left[\left(\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\partial u_3}\right)^2+\left(\frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3}\right)^2\right], \tag{47}$$

which by means of (46) can be written

$$\frac{1}{R_3^2}=\frac{\lambda^2}{\phi^2}.$$

Hence the necessary and sufficient condition that the curves of parameter u_3 be circles is that ϕ be of the form

$$\phi=\lambda\psi, \tag{47'}$$

where ψ is a function of u_1 and u_2. Expressing the fact that this value of ϕ must satisfy equations (21) and making use of equations (46) in the reduction, we get

$$\begin{aligned}\psi\frac{\partial\log\lambda}{\partial u_2}+\frac{\partial\psi}{\partial u_2}&=\left[F_2-\psi\int\sin\omega\frac{\partial\lambda}{\partial u_3}\,du_3\right]\sin\alpha,\\ \psi\frac{\partial\log\lambda}{\partial u_1}+\frac{\partial\psi}{\partial u_1}&=\left[F_1-\psi\int\cos\omega\frac{\partial\lambda}{\partial u_3}\,du_3\right]\cos\alpha.\end{aligned} \tag{48}$$

Differentiating with respect to u_3, we get for the equations to be satisfied by λ

$$\begin{aligned}\frac{\partial\omega}{\partial u_3}\frac{\partial^2\log\lambda}{\partial u_2\,\partial u_3}+\frac{\partial^2\omega}{\partial u_2\,\partial u_3}\frac{\partial\log\lambda}{\partial u_3}&=0,\\ \frac{\partial\omega}{\partial u_3}\frac{\partial^2\log\lambda}{\partial u_1\,\partial u_3}+\frac{\partial^2\omega}{\partial u_1\,\partial u_3}\frac{\partial\log\lambda}{\partial u_3}&=0,\end{aligned} \tag{49}$$

which are readily found to be compatible by means of equations (18). If the values of λ given by equations (46) are substituted in the above equations they are satisfied identically. Hence whenever ω satisfies (45) in addition to the fundamental equations (8), (18), and (19), it only remains to determine ψ from equations (48) and then the A-systems with the given representation and with circular curves of parameter u_3 are known. The compatibility of the system (25) of which (48) is a particular case, has been established and from the form of the latter it is seen that ψ can be determined by quadratures.

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* Bianchi, Lezioni, II, p. 498.

Equations (49) are satisfied identically when λ is independent of u_3. We shall consider this particular case at length. Now equations (18) reduce to

$$(50)\qquad \begin{aligned} \frac{\partial \omega}{\partial u_1} &= \lambda \cos\alpha \sin\omega + \frac{1}{\lambda \sin\alpha}\frac{\partial}{\partial u_2}\lambda\cos\alpha, \\ \frac{\partial \omega}{\partial u_2} &= -\left(\lambda \sin\alpha \cos\omega + \frac{1}{\lambda\cos\alpha}\frac{\partial}{\partial u_1}\lambda\sin\alpha\right), \end{aligned}$$

and equations (19) can be put in the form

$$\begin{aligned} \frac{\partial \omega}{\partial u_3}\left[\frac{1}{\cos\alpha}\frac{\partial \lambda}{\partial u_1} + \cos\omega\left(\lambda^2 - \frac{1}{U_3^2}\right)\right] &= \frac{\sin\omega}{U_3}\frac{\partial}{\partial u_3}\left(\frac{1}{U_3}\right), \\ \frac{\partial \omega}{\partial u_3}\left[\frac{1}{\sin\alpha}\frac{\partial \lambda}{\partial u_2} + \sin\omega\left(\lambda^2 - \frac{1}{U_3^2}\right)\right] &= -\frac{\cos\omega}{U_3}\frac{\partial}{\partial u_3}\left(\frac{1}{U_3}\right). \end{aligned}$$

Multiply these equations by $\cos\omega$ and $\sin\omega$ respectively and add; again by $\sin\omega$ and $-\cos\omega$ and add; this gives

$$(51)\qquad \begin{aligned} &\frac{\cos\omega}{\cos\alpha}\frac{\partial\lambda}{\partial u_1} + \frac{\sin\omega}{\sin\alpha}\frac{\partial\lambda}{\partial u_2} + \left(\lambda^2 - \frac{1}{U_3^2}\right) = 0, \\ &\frac{\sin\omega}{\cos\alpha}\frac{\partial\lambda}{\partial u_1} - \frac{\cos\omega}{\sin\alpha}\frac{\partial\lambda}{\partial u_2} - \frac{1}{\frac{\partial\omega}{\partial u_3}}\frac{1}{U_3}\frac{\partial}{\partial u_3}\left(\frac{1}{U_3}\right) = 0. \end{aligned}$$

Expressing the fact that ω, as given by (50), is a solution of (8), we are brought to the equation

$$(52)\qquad \begin{aligned} &\frac{\sin\omega}{\cos\alpha}\frac{\partial\lambda}{\partial u_1} + \frac{\cos\omega}{\sin\alpha}\frac{\partial\lambda}{\partial u_2} + \left(\lambda^2 - \frac{1}{U_3^2}\right)\sin\omega\cos\omega \\ &\quad + \frac{\partial}{\partial u_1}\left(\frac{1}{\lambda\sin\alpha}\frac{\partial}{\partial u_2}\lambda\cos\alpha\right) + \frac{\partial}{\partial u_2}\left(\frac{1}{\lambda\cos\alpha}\frac{\partial}{\partial u_1}\lambda\sin\alpha\right) = 0. \end{aligned}$$

Since λ is independent of u_3, for the expression $\lambda^2 - \frac{1}{U_3^2}$ to be zero, both λ and U_3 must be constant in which case equations (51) and (52) are satisfied. Assuming this function to be different from zero, we eliminate it from the first of (51) and (52) and get

$$\frac{\sin^3\omega}{\cos\alpha}\frac{\partial\lambda}{\partial u_1} + \frac{\cos^3\omega}{\sin\alpha}\frac{\partial\lambda}{\partial u_2} + \frac{\partial}{\partial u_1}\left(\frac{1}{\lambda\sin\alpha}\frac{\partial}{\partial u_2}\lambda\cos\alpha\right) + \frac{\partial}{\partial u_2}\left(\frac{1}{\lambda\cos\alpha}\frac{\partial}{\partial u_1}\lambda\sin\alpha\right) = 0,$$

which is an impossible equation as ω cannot be independent of u_3. Hence when λ is independent of u_3, it is a constant and the corresponding A-systems are Weingarten systems. Without loss of generality equations (46) can now be written

$$\begin{aligned}\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\,\partial u_3}&=\cos\alpha\frac{\partial\omega}{\partial u_3},\\ \frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\,\partial u_3}&=\sin\alpha\frac{\partial\omega}{\partial u_3},\end{aligned}\tag{53}$$

and equations (50) reduce to

$$\begin{aligned}\frac{\partial\omega}{\partial u_1}+\frac{\partial\alpha}{\partial u_2}&=\cos\alpha\sin\omega,\\ \frac{\partial\omega}{\partial u_2}+\frac{\partial\alpha}{\partial u_1}&=-\sin\alpha\cos\omega.\end{aligned}\tag{54}$$

Now the expression for R_3 reduces to

$$R_3=\phi,$$

with ϕ a function of u_1 and u_2. Conversely for ϕ to be a function of u_1 and u_2, it is necessary (27) that

$$\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\,\partial u_3}=f_1(u_1,u_2)\frac{\partial\omega}{\partial u_3},\qquad \frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\,\partial u_3}=f_2(u_1,u_2)\frac{\partial\omega}{\partial u_3},$$

where f_1 and f_2 are independent of u_3, which reduce as we have seen to $\cos\alpha$ and $\sin\alpha$ in (53). Hence the theorem:

Where ϕ is a function of u_1 and u_2 and independent of u_3, the curves of parameter u_3 are circles.

In arriving at this result we excluded the case where either u_1 or u_2 does not appear in ϕ. It was shown earlier that in this case the surfaces of two families of the system are moulure surfaces and the others are planes. To get the systems with ϕ independent of u_2 and u_3, in which case the curves of parameter u_3 are circles, it is only necessary to draw a family of circles in a plane and their orthogonal trajectories and rotate the plane about a cylinder.

It remains to consider in this connection the case where ϕ is a constant. We have seen that the surfaces $u_3=$ const. are moulure surfaces, which need not be considered in consequence of the above remarks, or pseudospherical surfaces. From (47′) it follows that in this case λ does not involve u_3, and consequently both λ and ψ are constants, so that R_3 is a constant. When equations (53) hold, the second of (34) is verified; and when the latter obtains, the curvature R_3 for the corresponding pseudospherical system is constant, so that Bianchi refers to such a system as a *system of Weingarten of constant curvature.* We have then the following theorem:

The only pseudospherical systems for which the orthogonal trajectories of the pseudospherical surfaces are circles are systems of Weingarten of constant curvature and all the circles are equal.

Now the equations (54) define the pseudospherical surfaces $u_3 = \text{const.}$ as the transforms of the pseudospherical surface whose linear element is

$$ds^2 = \cos^2\alpha \, du_1^2 + \sin^2\alpha \, du_2^2.$$

Hence the theorem:

Every Lame system of pseudospherical surfaces whose orthogonal trajectories are circles is composed of the Bianchi transforms of a pseudospherical surface.

We have shown* that all the transforms under angle $\frac{\pi}{2}$ of an A-surface S are arranged in such a manner with respect to S that the points on these transforms corresponding to a given point on S lie on the circumference of a circle whose axis is normal to S at the given point and consequently the circle cuts all the transforms orthogonally. Hence the transforms of an A-surface under angle $\frac{\pi}{2}$ form a family of Lame with circular orthogonal trajectories.

It has been shown† likewise that if an A-surface be transformed so that the tangent planes to the transforms make the same angle σ with the tangent plane to the given surface the points of contact of the former corresponding to the similar point for the latter lie on the circumference of a circle whose axis is normal to the given surface at the given point. It is evident that for this case the curves of parameter u_3 are not orthogonal to the surface, where we have denoted the transforms by $u_3 = \text{const.}$, unless σ is a right angle.

It remains to be seen whether there are families of transforms for σ a function of u_3 such that they form part of an A-system. From our previous discussion it follows that all the pseudospherical surfaces associated with these transforms are the transforms under the corresponding angles σ of the pseudospherical surface associated with the given A-surface. Moreover, the A-surfaces will not form part of an A-system unless the associated pseudospherical surfaces form such a system. It has been remarked that the points of contact of the transforms of a pseudospherical surface of whatever angle σ lie in the tangent plane to the given surface at the corresponding point. Hence for the system under consideration the curves of parameter u_3 lie in the tangent planes to the given pseudospherical surface and hence cut the tangent planes to the transforms under an angle different from a right angle. Hence we have the theorem:

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* Surfaces with the same etc. l. c. p. 138.

† l. c. p. 152.

The only family of Bäcklund transforms of an A-surface forming part of an A-system is the family for which σ is a right angle.

This result suggests the inquiry as to whether all A-systems with circular orthogonal trajectories and with ϕ a function of u_1 and u_2 are obtained in this way. For this case ω is given by (53) and (54), and equation (8) in which U_3 has been replaced by unity. Equations (54) give a transformation of angle $\alpha + \pi$ for each of the surfaces $u_3 =$ const. such that the lines of curvature on all these transforms have the same spherical images, the linear element of the sphere being

$$d\sigma^2 = \sin^2\alpha \, du_1^2 + \cos^2\alpha \, du_2^2.$$

The linear element of each of these transforms is of the form*

$$ds^2 = \left(\frac{\partial H_1}{\partial u_1} + H_2 \frac{\partial \omega}{\partial u_1} - \lambda \cos\alpha\right)^2 du_1^2 + \left(\frac{\partial H_2}{\partial u_2} - H_1 \frac{\partial \omega}{\partial u_2} - \lambda \sin\alpha\right) du_2^2,$$

where $\lambda = H_1 \cos\omega + H_2 \sin\omega$. It is evident that H_1, H_2 and ω are functions of u_3. But by means of (12) and (53) it is readily found that the coefficients in the above expression are independent of u_3, so that all the transforms coincide. Hence the theorem:

The A-systems for which ϕ is a function of u_1 and u_2 are composed of the transforms of an A-surface and their circular trajectories.

And since equations (53) are a consequence of (54) we have the converse theorem:

For all A-systems formed of the complementary transforms of an A-surface the function ϕ is independent of u_3.

§8. *A-Systems with the Curves of Parameter u_1 Plane.*

We have seen elsewhere† that the lines of curvature in both systems are plane for moulure surfaces and that there are certain A-surfaces on which the lines of curvature in one system are plane. *A*-Systems with a family of moulure surfaces have been considered so that we propose now a study of those *A*-systems for which the curves of parameter u_1 are plane. For this we must have*

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* l. c. p. 131.

† Certain Surfaces with Plane and Spherical Lines of Curvature, l. c. p. 50.

$$\frac{\partial \omega}{\partial u_2} = \mu \sin \omega\,, \tag{55}$$

where μ is independent of u_1, or the equivalent condition

$$\frac{\partial \omega}{\partial u_1} = \lambda \sin \omega\,, \tag{56}$$

where λ does not involve u_2. If these values be substituted in equation (8), the latter becomes

$$\cos \omega = \frac{\dfrac{\partial \mu}{\partial u_2} - \dfrac{\partial \lambda}{\partial u_1}}{\lambda^2 - \mu^2 - \dfrac{1}{U_3^2}}\,. \tag{57}$$

Eliminating ω from this equation and (55), we get

$$\left(\lambda^2 - \mu^2 - \frac{1}{U_3^2}\right)^2 - \left(\frac{\partial \mu}{\partial u_2}\right)^2 = \frac{\dfrac{\partial^2 \lambda}{\partial u_1^2}}{\lambda}\left(\lambda^2 - \mu^2 - \frac{1}{U_3^2}\right) - \left(\frac{\partial \lambda}{\partial u_1}\right)^2, \tag{58}$$

which upon differentiation with respect to u_2 becomes

$$\frac{\partial \mu}{\partial u_2}\left[\mu\, \frac{\partial^2 \lambda}{\partial u_1^2} \Big/ \frac{\partial \lambda}{\partial u_1} - 2\mu\left(\lambda^2 - \mu^2 - \frac{1}{U_3^2}\right) - \frac{\partial^2 \mu}{\partial u_2^2}\right] = 0.$$

If $\dfrac{\partial \mu}{\partial u_2}$ is equal to zero, it follows from (57) that ω does not involve u_2, and consequently we should be dealing with a system of moulure surfaces. Equating to zero the expression in the parenthesis and differentiating with respect to u_1, we get

$$\frac{\partial}{\partial u_1}\left(\frac{\dfrac{\partial^2 \lambda}{\partial u_1^2}}{\dfrac{\partial \lambda}{\partial u_1}}\right) - 4\lambda \frac{\partial \lambda}{\partial u_1} = 0.$$

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* Certain Surfaces with Plane and Spherical Lines of Curvature, l. c. p. 50.

Upon integration this becomes

$$\left(\frac{\partial \lambda}{\partial u_1}\right)^2 = \lambda^4 + 2a\lambda^2 + b,$$

where a and b are at most functions of u_3. If this value be substituted in (58), it becomes upon integration

$$(58') \qquad \left(\frac{\partial \mu}{\partial u_2}\right)^2 = \left(\mu^2 + \frac{1}{U_3^2}\right)^2 + 2a\left(\mu^2 + \frac{1}{U_3^2}\right) + b.$$

It is readily found that a and b cannot be zero at the same time.

For the second of equations (18) to be satisfied when ω is given by equations (55) and (56) it is necessary that λ be independent of u_3, hence a and b are constants in the above expression. Noting this fact, we make a similar substitution in the second of equations (19) and get

$$2\frac{\partial \omega}{\partial u_3}\frac{\partial \mu}{\partial u_2} + \sin\omega \frac{\partial}{\partial u_3}\left(\lambda^2 - \mu^2 - \frac{1}{U_3^2}\right) = 0.$$

If ω be eliminated from this equation and (57), the result is

$$\left[\left(\lambda^2 - \mu^2 - \frac{1}{U_3^2}\right)^2 + \left(\frac{\partial \mu}{\partial u_2}\right)^2 + \left(\frac{\partial \lambda}{\partial u_1}\right)^2\right]\frac{\partial}{\partial u_3}\left(\lambda^2 - \mu^2 - \frac{1}{U_3^2}\right)$$
$$- \left(\lambda^2 - \mu^2 - \frac{1}{U_3^2}\right)\frac{\partial}{\partial u_3}\left(\frac{\partial \mu}{\partial u_2}\right)^2 = 0,$$

which equation is found to be satisfied identically when $\frac{\partial \lambda}{\partial u_1}$ and $\frac{\partial \mu}{\partial u_2}$ are replaced by their above expressions. Hence given two solutions λ and μ of the above equations in which a and b are arbitrary constants and U_3 is an arbitrary function of u_3, then ω given by (57) determines the spherical representation of a whole group of A-systems in which the curves of parameter u_1 are plane.

By means of (56) the second of equations (21) can be reduced to

$$(59) \qquad \frac{\partial \phi}{\partial u_1} = \left(F_1 - \int \cos\omega \frac{\partial \phi}{\partial u_3}\, du_3\right)\lambda.$$

If this equation and (56) be differentiated with respect to u_3 and $\cos\omega$ eliminated from the resulting equations, one gets

$$\frac{\partial \omega}{\partial u_3}\frac{\partial^2 \phi}{\partial u_1 \partial u_3} + \frac{\partial \phi}{\partial u_3}\frac{\partial^2 \omega}{\partial u_1 \partial u_3} = 0,$$

of which the general integral has the form

$$\phi = \int \frac{f(u_2, u_3)}{\frac{\partial \omega}{\partial u_3}} du_3 + \psi(u_1, u_2),$$

where f is a function independent of u_1, and ψ does not involve u_3. When this value is substituted in (59), it is found that

$$F_1 \lambda = \frac{\partial \psi}{\partial u_1}$$

and from the second of equations (20) it is found that F_2 is a function of u_2 alone. Since ϕ must satisfy the third of equations (27), one finds at once that the function $f(u_2, u_3)$ is an integral of a partial differential equation of the second order of the Laplace type. The functions ψ and F_2 are determined by the equations of condition (20); it is readily seen that the equation for the former is reducible to the form of an ordinary linear equation of the first order. Later we shall find particular A-systems of the kind which we have here considered in general.

In the A-system of which the surfaces $u_3 =$ const. are moulure surfaces the curves of parameter u_1 and of parameter u_3 are plane. We inquire whether there are any other A-systems having this property. Now ω must satisfy equations (46), (55), and (56), of which, as we have seen, the last can be written

$$\frac{1}{\sin \omega} \frac{\partial \omega}{\partial u_1} = U_1,$$

where U_1 is a function of u_1 alone. If this value of $\frac{\partial \omega}{\partial u_1}$ be substituted in the first of (46), one gets

$$\lambda \cos \alpha = U_1,$$

from which it follows that λ is independent of u_3 and consequently a constant; and moreover α is a function of u_1 alone. Again, since λ is constant, the system is a Weingarten system and hence from (58′) we have that μ does not involve u_3. From the second of (46) and (55), we get

$$\mu = \lambda \sin \alpha \tan \omega,$$

which is an impossible equation, since ω must involve u_3. Hence

Moulure A-systems are the only ones in which the orthogonal trajectories in two systems are plane curves.

§9. *A-Systems with the Curves of Parameter u_3 of Constant Curvature.*

We have met with pseudospherical systems for which all the curves of parameter u_3 have constant principal curvature. We seek now all A-systems which have this property and without loss of generality can take the radius of principal curvature equal to unity.

For this case we have from (47)

$$\left(\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\partial u_3}\right)^2+\left(\frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3}\right)^2=\phi^2\left(\frac{\partial\omega}{\partial u_3}\right). \tag{60}$$

In accordance with this relation we introduce an auxiliary function α defined by

$$\begin{aligned}\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\partial u_3}&=\phi\cos\alpha\frac{\partial\omega}{\partial u_3},\\ \frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3}&=\phi\sin\alpha\frac{\partial\omega}{\partial u_3}.\end{aligned} \tag{61}$$

In consequence of these relations equations (18) can be put in the form

$$\left\{\begin{aligned}\frac{\partial\phi}{\partial u_1}&=-\phi\cot\alpha\left(\frac{\partial\alpha}{\partial u_1}+\frac{\partial\omega}{\partial u_2}\right)-\phi^2\cos\alpha\cos\omega,\\ \frac{\partial\phi}{\partial u_2}&=\phi\tan\alpha\left(\frac{\partial\alpha}{\partial u_2}+\frac{\partial\omega}{\partial u_1}\right)-\phi^2\sin\alpha\sin\omega;\end{aligned}\right. \tag{62}$$

and in place of (19) we have

$$\left\{\begin{aligned}\frac{\partial\phi}{\partial u_1}&=\phi\tan\alpha\left(\frac{\partial\alpha}{\partial u_1}+\frac{\partial\omega}{\partial u_2}\right)-\phi^2\cos\alpha\cos\omega+\frac{\dfrac{\partial}{\partial u_3}\left(\dfrac{\sin\omega}{U_3}\right)}{U_3\cos\alpha\dfrac{\partial\omega}{\partial u_3}},\\ \frac{\partial\phi}{\partial u_2}&=-\phi\cot\alpha\left(\frac{\partial\alpha}{\partial u_2}+\frac{\partial\omega}{\partial u_1}\right)-\phi^2\sin\alpha\sin\omega-\frac{\dfrac{\partial}{\partial u_3}\left(\dfrac{\cos\omega}{U_3}\right)}{U_3\sin\alpha\dfrac{\partial\omega}{\partial u_3}}.\end{aligned}\right. \tag{63}$$

Equating the above expressions for $\dfrac{\partial\phi}{\partial u_1}$ and $\dfrac{\partial\phi}{\partial u_2}$, we get

$$\left\{\begin{aligned}&\phi\left(\frac{\partial\alpha}{\partial u_1}+\frac{\partial\omega}{\partial u_2}\right)+\frac{\sin\alpha\dfrac{\partial}{\partial u_3}\left(\dfrac{\sin\omega}{U_3}\right)}{U_3\dfrac{\partial\omega}{\partial u_3}}=0,\\ &\phi\left(\frac{\partial\alpha}{\partial u_3}+\frac{\partial\omega}{\partial u_1}\right)+\frac{\cos\alpha\dfrac{\partial}{\partial u_3}\left(\dfrac{\cos\omega}{U_3}\right)}{U_3\dfrac{\partial\omega}{\partial u_3}}=0.\end{aligned}\right. \tag{64}$$

It is readily shown that these equations are consistent when the former conditions are satisfied. By means of these relations the equations (62) can be put in the form

$$\frac{\partial \phi}{\partial u_1} = \frac{\cos \alpha \dfrac{\partial}{\partial u_3} \dfrac{\sin \omega}{U_3}}{U_3 \dfrac{\partial \omega}{\partial u_3}} - \phi^2 \cos \alpha \cos \omega,$$

$$\frac{\partial \phi}{\partial u_2} = - \frac{\sin \alpha \dfrac{\partial}{\partial u_3} \dfrac{\cos \omega}{U_3}}{U_3 \dfrac{\partial \omega}{\partial u_3}} - \phi^2 \sin \alpha \sin \omega\,;$$

and these equations are consistent from what precedes. When these values are substituted in the first of equations (27), the resulting equation can be reduced to

$$(65) \qquad \begin{aligned} &\left(\frac{\partial}{\partial u_3} \log \frac{1}{U_3}\right)^2 + 2 \cot 2\omega \frac{\partial \omega}{\partial u_3} \frac{\partial}{\partial u_3} \log \frac{1}{U_3} \\ &\quad + U_3^2 \left[\left(\frac{1}{\cos \omega} \frac{\partial^2 \omega}{\partial u_1 \partial u_3}\right)^2 + \left(\frac{1}{\sin \omega} \frac{\partial^2 \omega}{\partial u_2 \partial u_3}\right)^2 - \left(\frac{\partial \omega}{\partial u_3}\right)^2 \right] = 0. \end{aligned}$$

Differentiate this equation with respect to u_1; the result can be reduced by means of (18) and (19) to

$$\left(\sin \omega \cos \omega \frac{\partial^2 \omega}{\partial u_1 \partial u_3} - \frac{\partial \omega}{\partial u_1} \frac{\partial \omega}{\partial u_3}\right) \frac{\partial U_3}{\partial u_3} = 0.$$

In like manner differentiation with respect to u_2 leads to

$$\left(\sin \omega \cos \omega \frac{\partial^2 \omega}{\partial u_2 \partial u_3} + \frac{\partial \omega}{\partial u_2} \frac{\partial \omega}{\partial u_3}\right) \frac{\partial U_3}{\partial u_3} = 0.$$

Unless U_3 is a constant, the quantities in parenthesis are zero. For the latter case we have upon integration with respect to u_3

$$\frac{\partial \omega}{\partial u_1} = a \tan^2 \omega, \qquad \frac{\partial \omega}{\partial u_2} = b \cot^2 \omega,$$

where a and b are independent of u_3. For these equations to be compatible, we must have

$$\frac{\partial a}{\partial u_2} \tan^4 \omega + 4ab \tan \omega \sec^2 \omega - \frac{\partial b}{\partial u_1} = 0,$$

which is impossible, for ω must depend upon u_3. Hence U_3 is a constant and (65) becomes

$$\left(\frac{1}{\cos\omega}\,\frac{\partial^2\omega}{\partial u_1\,\partial u_3}\right)^2+\left(\frac{1}{\sin\omega}\,\frac{\partial^2\omega}{\partial u_2\,\partial u_3}\right)^2-\left(\frac{\partial\omega}{\partial u_3}\right)^2=0,$$

so that ϕ is equal to unity in (60). Hence the theorem:

Pseudospherical systems of Weingarten of constant curvature are the only A-systems for which the orthogonal trajectories of the A-surfaces have constant principal curvature.

§10. *A-Systems with Spherical Lines of Parameter u_1.*

In a former paper* we have considered *A*-surfaces whose lines of curvature in one family are spherical. We shall apply certain of the results obtained at that time in considering *A*-systems for which the curves of parameter u_1 are spherical.

One must have†

$$H_1=\alpha\sin\omega+\beta\,\frac{\partial\omega}{\partial u_2}, \tag{66}$$

where α and β are functions of u_2 and u_3. From the first of (7) it follows that

$$H_2=-\left(\alpha\cos\omega+\frac{\partial\beta}{\partial u_2}+\frac{\sin\omega\,\dfrac{\partial\alpha}{\partial u_2}+\beta\,\dfrac{\partial^2\omega}{\partial u_2^2}}{\dfrac{\partial\omega}{\partial u_2}}\right), \tag{67}$$

and the second of (7) may be written

$$\begin{aligned}&\left(\cos\omega\,\frac{\partial\omega}{\partial u_1}\,\frac{\partial\omega}{\partial u_2}-\sin\omega\,\frac{\partial^2\omega}{\partial u_1\,\partial u_2}\right)\frac{\partial\alpha}{\partial u_2}\\ &\qquad+\left[\frac{\partial\omega}{\partial u_2}\,\frac{\partial^3\omega}{\partial u_1\,\partial u_2^2}-\frac{\partial^2\omega}{\partial u_2^2}\,\frac{\partial^2\omega}{\partial u_1\,\partial u_2}+\frac{\partial\omega}{\partial u_1}\left(\frac{\partial\omega}{\partial u_2}\right)^3\right]\beta=0.\end{aligned} \tag{68}$$

Differentiate this equation with respect to u_1, and eliminate $\dfrac{\partial\alpha}{\partial u_2}$ and β from the resulting equation and (68); this gives

$$\begin{vmatrix}\cos\omega\,\dfrac{\partial\omega}{\partial u_1}\,\dfrac{\partial\omega}{\partial u_2}-\sin\omega\,\dfrac{\partial^2\omega}{\partial u_1\,\partial u_2} & \dfrac{\partial\omega}{\partial u_2}\,\dfrac{\partial^3\omega}{\partial u_1\,\partial u_2^2}-\dfrac{\partial^2\omega}{\partial u_2^2}\,\dfrac{\partial^2\omega}{\partial u_1\,\partial u_2}+\dfrac{\partial\omega}{\partial u_1}\left(\dfrac{\partial\omega}{\partial u_2}\right)^3\\ \partial\Big/\partial u_1\left[\cos\omega\,\dfrac{\partial\omega}{\partial u_1}\,\dfrac{\partial\omega}{\partial u_2}-\sin\omega\,\dfrac{\partial^2\omega}{\partial u_1\,\partial u_2}\right] & \dfrac{\partial}{\partial u_1}\left[\dfrac{\partial\omega}{\partial u_2}\,\dfrac{\partial^3\omega}{\partial u_1\,\partial u_2^2}-\dfrac{\partial^2\omega}{\partial u_2^2}\,\dfrac{\partial^2\omega}{\partial u_1\,\partial u_2}+\dfrac{\partial\omega}{\partial u_1}\left(\dfrac{\partial\omega}{\partial u_2}\right)^3\right]\end{vmatrix}=0, \tag{69}$$

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*Certain Surfaces with Plane or Spherical Lines of Curvature, l. c. p. 60.

†Ib., p. 61.

an equation which ω must satisfy. From the first of equations (11) we get

$$H_3 = -U_3\left[\frac{\partial\alpha}{\partial u_3} + \alpha\cot\omega\frac{\partial\omega}{\partial u_3} + \frac{1}{\sin\omega}\frac{\partial\omega}{\partial u_2}\frac{\partial\beta}{\partial u_3} + \frac{\beta}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\,\partial u_3}\right]. \tag{70}$$

If these expressions for H_1 and H_3 be substituted in equation (16), it becomes

$$\left[\cos\omega\frac{\partial\omega}{\partial u_1}\frac{\partial\omega}{\partial u_2} - \sin\omega\frac{\partial^2\omega}{\partial u_1\,\partial u_2}\right]\frac{\partial\beta}{\partial u_3} - \left[\tan\omega\frac{\partial^2\omega}{\partial u_1\,\partial u_3} - \frac{\partial\omega}{\partial u_1}\frac{\partial\omega}{\partial u_3}\right]\alpha = 0. \tag{71}$$

Differentiate this equation with respect to u_1 and eliminate α and $\frac{\partial\beta}{\partial u_3}$; this gives the following additional condition for ω

$$\begin{aligned}&\left(\cos\omega\frac{\partial\omega}{\partial u_1}\frac{\partial\omega}{\partial u_2} - \sin\omega\frac{\partial^2\omega}{\partial u_1\,\partial u_2}\right)\left[\frac{\partial\omega}{\partial u_1}\frac{\partial^3\omega}{\partial u_1^2\,\partial u_3} - \frac{\partial^2\omega}{\partial u_1^2}\frac{\partial^2\omega}{\partial u_1\,\partial u_3}\right.\\ &\qquad\qquad\qquad\qquad\qquad\qquad\left. + \tan\omega\left(\frac{\partial\omega}{\partial u_1}\right)^2\frac{\partial^2\omega}{\partial u_1\,\partial u_3}\right]\\ &= \left(\cos\omega\frac{\partial\omega}{\partial u_1}\frac{\partial\omega}{\partial u_3} - \sin\omega\frac{\partial^2\omega}{\partial u_1\,\partial u_3}\right)\left[\frac{\partial\omega}{\partial u_1}\frac{\partial^3\omega}{\partial u_1^2\,\partial u_3} - \frac{\partial^2\omega}{\partial u_1^2}\frac{\partial^2\omega}{\partial u_1\,\partial u_2} + \left(\frac{\partial\omega}{\partial u_1}\right)^3\frac{\partial\omega}{\partial u_2}\right].\end{aligned} \tag{72}$$

It is readily seen that if ω satisfies (69) and (72) in addition to the fundamental equations (8), (18), (19), it remains for H_1 and H_2 to satisfy (12), which by means of (68) and (71) can be brought to the form

$$\begin{aligned}&\beta\sin\omega\frac{\partial\omega}{\partial u_2}\frac{\partial\Omega}{\partial u_2} - \Omega\left[\alpha\tan\omega\frac{\partial\omega}{\partial u_2} + \frac{\partial\alpha}{\partial u_2}\sin^2\omega\right.\\ &\qquad\qquad\left. + \beta\left(\sin\omega\frac{\partial^2\omega}{\partial u_2^2} + \frac{\sin^2\omega}{\cos\omega}\left(\frac{\partial\omega}{\partial u_2}\right)^2\right)\right] = 0,\end{aligned} \tag{73}$$

where

$$\Omega = \frac{\partial^3\omega}{\partial u_1\partial u_2\partial u_3} - \cot\omega\frac{\partial\omega}{\partial u_1}\frac{\partial^2\omega}{\partial u_2\partial u_3} + \tan\omega\frac{\partial\omega}{\partial u_2}\frac{\partial^2\omega}{\partial u_2\partial u_3}.$$

By means of (68) and a differentiation with respect to u_1, the functions α, β, and $\frac{\partial\alpha}{\partial u_2}$ can be eliminated from (73) thus giving another equation of condition for ω. When all of these conditions are satisfied equations (68) and (71) can be written

$$\frac{\partial\beta}{\partial u_3} = \alpha\phi_1, \qquad \frac{\partial\alpha}{\partial u_2} = \beta\phi_2, \tag{74}$$

where ϕ_1 and ϕ_2 are known functions, unless the lines $u_2 = \text{const.}$ are plane, and equation (73) is of the form

$$\lambda \frac{\partial \alpha}{\partial u_2} + \mu \alpha = 0,$$

with λ and μ known functions. Hence unless λ is zero, α is given by a quadrature and β directly from the second of (74), but the values so found must satisfy the first of these equations. When λ is zero, the determination of α requires the integration of a differential equation of the second order and β follows directly. Several particular cases of the foregoing problem can be solved and are of interest.

Consider first the case where

$$\frac{\partial \omega}{\partial u_1} \frac{\partial^3 \omega}{\partial u_1^2 \partial u_3} - \frac{\partial^2 \omega}{\partial u_1^2} \frac{\partial^2 \omega}{\partial u_1 \partial u_3} + \tan \omega \left(\frac{\partial \omega}{\partial u_1} \right)^2 \frac{\partial^2 \omega}{\partial u_1 \partial u_3} = 0,$$

which by integration with respect to u_1 becomes

$$(75) \qquad \frac{1}{\cos \omega} \frac{\partial^2 \omega}{\partial u_1 \partial u_3} = \rho \frac{\partial \omega}{\partial u_1},$$

where ρ is a function independent of u_1. If this value be substituted in the first of equations (18), one gets

$$\frac{1}{\sin \omega} \frac{\partial^2 \omega}{\partial u_2 \partial u_3} \cdot \frac{\partial \omega}{\partial u_1} = \frac{\partial \omega}{\partial u_1} \frac{\partial \rho}{\partial u_2} + \rho \frac{\partial^2 \omega}{\partial u_1 \partial u_2}.$$

For these values the second of equations (18) becomes

$$\frac{\partial \omega}{\partial u_1} \frac{\partial^3 \omega}{\partial u_1^2 \partial u_2} - \frac{\partial^2 \omega}{\partial u_1^2} \frac{\partial^2 \omega}{\partial u_1 \partial u_2} + \left(\frac{\partial \omega}{\partial u_1} \right)^2 \frac{\partial \omega}{\partial u_2} = 0.$$

Then equation (72) is satisfied and also (69), as has been seen before. It is readily found that for this case Ω is zero. Hence when ω is such that the ratio of $\frac{1}{\cos \omega} \frac{\partial^2 \omega}{\partial u_1 \partial u_3}$ and $\frac{\partial \omega}{\partial u_1}$ is independent of u_1, there is a double infinity of *A*-systems for which the curves of parameter u_1 are spherical. From the investigation* of pseudospherical surfaces with spherical lines of curvature it follows that for the pseudospherical systems associated with the above *A*-systems the curves of parameter u_1 are spherical.

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* Certain Surfaces with Plane or Spherical Lines of Curvature, l. c., p. 62.

If we have

$$\cos\omega\frac{\partial\omega}{\partial u_1}\frac{\partial\omega}{\partial u_2}-\sin\omega\frac{\partial^2\omega}{\partial u_1\partial u_2}=0, \tag{76}$$

that is, if the lines $u_2=$ const. on the surfaces $u_3=$ const. are plane, and this equation is differentiated with respect to u_3, it can be reduced by means of (18) to

$$\cos\omega\frac{\partial\omega}{\partial u_1}\frac{\partial\omega}{\partial u_3}-\sin\omega\frac{\partial^2\omega}{\partial u_1\partial u_3}=0. \tag{77}$$

By means of (76) it can be shown that this equation satisfies the condition (75). Moreover, the above equations can be written

$$\frac{\partial}{\partial u_3}\left(\frac{1}{\sin\omega}\frac{\partial\omega}{\partial u_1}\right)=0, \qquad \frac{\partial}{\partial u_2}\left(\frac{1}{\sin\omega}\frac{\partial\omega}{\partial u_1}\right)=0,$$

from which it follows that

$$\tan\frac{\omega}{2}=F(u_1)+\psi(u_2,u_3), \tag{78}$$

where F involves only u_1 and ψ is independent of this parameter. From this we have

$$\frac{1}{\sin\omega}\frac{\partial\omega}{\partial u_2}=\frac{\partial\psi}{\partial u_2}, \tag{79}$$

and recalling the discussion for curves $u_2=$ const. plane, we remark that ψ must satisfy the condition

$$\left(\frac{\partial^2\psi}{\partial u_2^2}\right)^2=\left[\left(\frac{\partial\psi}{\partial u_2}\right)^2+\frac{1}{U_3^2}\right]^2+\left[\left(\frac{\partial\psi}{\partial u_2}\right)^2+\frac{1}{U_3^2}\right]a+b,$$

where a and b are arbitrary constants which cannot be zero simultaneously. Again we have

$$\frac{1}{\sin\omega}\frac{\partial\omega}{\partial u_1}=F',$$

so that for the determination of F we have the equation

$$F''^2=F'^4+2aF'^2+b.$$

Enneper* has given the first integral of this equation in the form

$$F'=\sqrt{a-A\cos h\,2F+B\sin h\,2F},$$

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* Göttingen, Nachrichten, 1868, p. 258.

where A and B are constants bound by the relation

$$B^2 - A^2 = b + 3a^2.$$

For this value of ω equations (68), (71), (73) vanish identically so that α and β are arbitrary functions of u_2 and u_3. From (79) it follows that the expression for H_1 can be reduced to

$$H_1 = \sin\omega\left(\alpha + \beta\frac{\partial\psi}{\partial u_2}\right),$$

consequently the linear element of space for these systems can be written in the form

$$ds^2 = a^2\sin^2\omega\, du_1^2 + \left(a\cos\omega + \frac{\sin\omega}{\frac{\partial\omega}{\partial u_2}}\frac{\partial a}{\partial u_2}\right)^2 du_2^2 + U_3^2\left[\frac{\partial a}{\partial u_3} + a\cot\omega\frac{\partial\omega}{\partial u_3}\right]^2 du_3^2.$$

Hence we have the theorem:

A function ω given by (78) *determines the spherical representation of an infinity of A-systems for which the curves of parameter u_1 are circles.*

§11. *A-Systems with the Surfaces $u_2 =$ const. Spheres.*

From the definition of the function α in (66)[*] it follows that the vanishing of α is the necessary and sufficient condition that the spheres, upon which the curves of parameter u_1 lie, cut the surface orthogonally. From the theorem of Darboux[†] that, when the surfaces in two families cut one another orthogonally in the lines of curvature of both they form part of a triply orthogonal system, it follows that the vanishing of α is the necessary and sufficient condition that the surfaces $u_2 =$ const. are spheres.

For α to be zero it is necessary in (68) that

$$(80)\qquad \frac{\partial\omega}{\partial u_2}\frac{\partial^3\omega}{\partial u_1\partial u_2^2} - \frac{\partial^2\omega}{\partial u_2^2}\frac{\partial^2\omega}{\partial u_1\partial u_2} + \frac{\partial\omega}{\partial u_1}\left(\frac{\partial\omega}{\partial u_2}\right) = 0$$

and from (71) it is seen that either β must be independent of u_3 or the lines $u_2 =$ const. be plane on the surfaces $u_3 =$ const. Let β be independent of u_3; the only other equation of condition to be satisfied is (12) which reduces to

$$\frac{\partial\omega}{\partial u_2}\frac{\partial^3\omega}{\partial u_2^2\partial u_3} - \frac{\partial^2\omega}{\partial u_2^2}\frac{\partial^2\omega}{\partial u_2\partial u_3} - \cot\omega\frac{\partial^2\omega}{\partial u_2\partial u_3} = 0,$$

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[*] l. c., p. 60.

[†] Bianchi, Lezioni, p. 454.

and an integration with respect to u_2 gives

$$(81) \qquad \frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\,\partial u_3} = \sigma\frac{\partial\omega}{\partial u_2},$$

where σ is a function independent of u_2. Proceeding with (81) in a manner analogous to the treatment of equation (75), we are brought to (80). We have then the theorem:

When the spherical representation of an A-system is such that the ratio of $\frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\,\partial u_3}$ *and* $\frac{\partial\omega}{\partial u_2}$ *is independent of* u_2, *there exist an infinity of A-systems with the same spherical representation and the surfaces* $u_2 = const.$ *are spheres.*

The linear element of these systems is reducible to

$$(82) \qquad ds^2 = \left(U_2\frac{\partial\omega}{\partial u_2}\right)^2 du_1^2 + \left(\frac{U_2\dfrac{\partial^2\omega}{\partial u_2^2}}{\dfrac{\partial\omega}{\partial u_2}} + U_2^1\right)^2 du_2^2 + U_1^2U_3^2\left(\frac{1}{\sin\omega}\,\frac{\partial^2\omega}{\partial u_2\partial u_3}\right)^2 du_3^2,$$

where U_2 is an arbitrary function of u_2 alone and the prime denotes differentiation. From the expression for the principal radius ρ_{21} in (4) it is found that ρ_{21} is equal to U_2. Consequently when U_2 is given a constant value all the spheres have the same radius.

It can be shown that the condition that the lines of parameter u_2 be plane, namely $\frac{\partial}{\partial u_2}\left(\frac{1}{\cos\omega}\,\frac{\partial\omega}{\partial u_1}\right) = 0$, satisfies (81) in a manner similar to that followed in the discussion of equation (75), and the form of ω can be determined in an analogous manner. Hence:

Among all the A-systems with the lines of parameter u_2 *plane and having the same spherical representation, there is an infinity for which the surfaces* $u_3 = const.$ *are spheres.*

The linear element of these systems is reducible to (82). For this case

$$\frac{1}{\cos\omega}\,\frac{\partial\omega}{\partial u_2} = f(u_2),$$

where f is a function of u_2 alone. When U_2 in (82) is replaced by the reciprocal of f, the linear element becomes

$$ds^2 = \cos^2\omega\,du_1^2 + \sin^2\omega\,du_2^2 + U_3^2\left(\frac{\partial\omega}{\partial u_3}\right)du_3^2,$$

that is, the corresponding system is pseudospherical. Bianchi has considered* these systems for U_3 a constant and has shown that the surfaces $u_3 = \text{const.}$ are surfaces of Enneper with the same axis. When in particular U_2 is a constant, these surfaces $u_3 = \text{const.}$ are surfaces of Dini; they have been discussed by Bianchi.†

Having thus considered the case where β is independent of u_3 in (71), it remains to discuss the case where the lines of parameter u_1 are plane. Since ρ_{21} must be a function of u_2 alone, it follows for this case also that β is a function of u_2 alone. Hence:

Among the A-systems having the same spherical representation and with the lines of parameter u_1 circular, there is an infinity for which the surfaces $u_2 = \text{const.}$ are spheres.

§12. *A-Systems with One Family of Surfaces of Bianchi.*

We have shown‡ that among a group of *A*-surfaces with the same spherical representation there are certain surfaces which have the property that the spheres described on the segments of the normals comprised between the centers of curvature as diameters cut a fixed sphere in great circles, or orthogonally, or all pass through the center; these we have called the *surfaces of Bianchi* of the *elliptic, hyperbolic,* and *parabolic* types respectively. We seek for *A*-systems in which all the surfaces in one family are surfaces of Bianchi of one of the three types.

Consider a pseudospherical surface S with the linear element

$$ds^2 = \cos^2\omega\, du_1^2 + \sin^2\omega\, du_2^2, \tag{83}$$

where ω is a particular solution of the last of equations (6). By X_1, Y_1, Z_1; X_2, Y_2, Z_2 we denote the direction-cosines of the directions of the lines of curvature $u_2 = \text{const.}$ and $u_1 = \text{const.}$ on this surface, and we define two functions α and β by the equations

$$\left\{\begin{aligned} \frac{\partial \alpha}{\partial u_1} &= \cos\theta\cos\omega\ \ , & \frac{\partial \alpha}{\partial u_2} &= \sin\theta\sin\omega, \\ \frac{\partial \beta}{\partial u_1} &= e^{\alpha}\sin\theta\cos\omega, & \frac{\partial \beta}{\partial u_2} &= -e^{\alpha}\cos\theta\sin\omega, \end{aligned}\right. \tag{84}$$

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[*] Annali, vol. 14, p. 115.

[†] Annali, vol. 13, p. 52

[‡] Surfaces with the same etc., l. c. p. 130.

when θ is a solution of the system (34′). Following the method of Bianchi we found that a surface of the parabolic type, Σ, with the spherical representation

$$d\sigma^2 = \cos^2\theta\, du_1^2 + \sin^2\theta\, du_2^2$$

is given by

$$\xi = (e^\alpha \cos\theta + \beta \sin\theta) X_1 + (e^\alpha \sin\theta - \beta\cos\theta) X_2 \tag{85}$$

and analogous expressions for η and ζ. The linear element is

$$ds^2 = (e^\alpha \cos\theta + \beta \sin\theta)\, du_1^2 + (e^\alpha \sin\theta - \beta\cos\theta)\, du_2^2. \tag{86}$$

Each solution θ of equations (34′) gives a corresponding surface Σ and now we inquire under what conditions all of these surfaces constitute a family of Lame. It is clear that the pseudospherical surfaces having the same spherical representation as these surfaces are the complimentary transforms of the surface S by means of the function θ. Consequently this function satisfies the equations of condition (8), (18), and (19), and since H_1 and H_2 in (86) satisfy equations (7), the only remaining condition is (12), which in consequence of (84) can be reduced to

$$e^\alpha \frac{\partial \alpha}{\partial u_3} + \beta \frac{\partial \theta}{\partial u_3} = 0. \tag{87}$$

From this it is seen that, when α is known, β is given directly and it is readily shown that this value satisfies (84). Since ω in (83) is independent of u_3, equations (34′) become upon differentiation with respect to u_3,

$$\begin{aligned} \frac{1}{\cos\theta} \frac{\partial^2\theta}{\partial u_1 \partial u_3} &= \cos\omega \frac{\partial \theta}{\partial u_3}, \\ \frac{1}{\sin\theta} \frac{\partial^2\theta}{\partial u_2 \partial u_3} &= \sin\omega \frac{\partial \theta}{\partial u_3}, \end{aligned} \tag{88}$$

so that the first of equations (84) may be replaced by

$$\frac{\partial \alpha}{\partial u_1} = \frac{\dfrac{\partial^2\theta}{\partial u_1 \partial u_3}}{\dfrac{\partial \theta}{\partial u_3}}, \qquad \frac{\partial \alpha}{\partial u_2} = \frac{\dfrac{\partial^2\theta}{\partial u_2 \partial u_3}}{\dfrac{\partial \theta}{\partial u_3}},$$

from which it follows that

$$e^\alpha = U_3 \frac{\partial \theta}{\partial u_3},$$

when U_3 is an arbitrary function of u_3. Then from (87) we have

$$\beta = -\left[\frac{U_3\dfrac{\partial^2\theta}{\partial u_3^2}}{\dfrac{\partial\theta}{\partial u_3}} + U_3'\right],$$

where the prime denotes differentiation. By means of (11) the coefficient H_3 is readily found, so that we get for the linear element of the system

$$(89)\quad ds^2 = (e^\alpha\cos\theta + \beta\sin\theta)^2\,du_1^2 + (e^\alpha\sin\theta - \beta\cos\theta)^2\,du_2^2 + \left(e^\alpha\frac{\partial\theta}{\partial u_3} - \frac{\partial\beta}{\partial u}\right)^2 du_3^2.$$

From (88) it follows that the lines of parameter u_3 in this system are plane curves.

We consider now a pseudospherical system of Weingarten with the linear element

$$(90)\qquad ds^2 = \cos^2\omega\,du_1^2 + \sin^2\omega\,du_2^2 + \left(\frac{\partial\omega}{\partial u_3}\right)^2 du_3^2,$$

and inquire under what conditions each of the pseudospherical surfaces admits a complimentary transform with which is associated a surface of Bianchi of the parabolic type in such a way that the latter surfaces form a family of Lame. From the results of the general discussion of Bäcklund transformations it follows that θ for the transformed system must be a solution of equations (34′) and (30) of which the latter reduces to

$$(91)\qquad \frac{\cos\theta}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\,\partial u_3} + \frac{\sin\theta}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\,\partial u_3} + \frac{\partial\omega}{\partial u_3} = 0.$$

We have remarked that such a solution is impossible when $F(u_3)$ in (33) is negative and subsequent developments will show that for the case under discussion $F(u_3)$ cannot vanish. Moreover, it was seen that θ so determined satisfies equations (8), (18), and (19), so that only (7) and (12) remain to be considered. But the former is satisfied by

$$H_1 = e^\alpha\cos\theta + \beta\sin\theta,\qquad H_2 = e^\alpha\sin\theta - \beta\cos\theta,$$

and then the latter reduces to (87). Differentiate the latter with respect to u_2; by means of (34′) and (84) this can be reduced to

$$(92)\qquad \left(\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\,\partial u_3} + \cos\theta\frac{\partial\omega}{\partial u_3}\right)\beta - e^\alpha\sin\theta\frac{\partial\omega}{\partial u_3} = 0.$$

Eliminating β between this equation and (87), we get

$$(93)\qquad \frac{\partial\alpha}{\partial u_3}\left[\frac{1}{\cos\omega}\,\frac{\partial^2\omega}{\partial u_1\,\partial u_3}+\cos\theta\,\frac{\partial\omega}{\partial u_3}\right]+\sin\theta\,\frac{\partial\omega}{\partial u_3}\,\frac{\partial\theta}{\partial u_3}=0.$$

If the expression in parenthesis were zero, equation (91) would reduce to

$$\frac{1}{\sin\omega}\,\frac{\partial^2\omega}{\partial u_2\,\partial u_3}-\sin\theta\,\frac{\partial\omega}{\partial u_3}=0,$$

and consequently the original pseudospherical system would be one of constant curvature, which has been excluded. Hence α is given by (84) and (93) to within an additive constant and β directly by (87). All the conditions of integrability are found to be satisfied and consequently we have a family of surfaces of Bianchi of the parabolic type with the linear element (89) where α and β have the above values. The above results were obtained by Bianchi[*] by a straightforward treatment of triply orthogonal systems, but we have seen fit to deduce them in a different manner as an example of the general theory of transformations of A-systems and especially as they furnish the basis of an investigation of the possible existence of systems of surfaces of Bianchi of the elliptic and hyperbolic types.

In order to make clear our procedure it is necessary to recall a theorem of Bianchi:[†] *If one takes any surface Σ of the parabolic type and a sphere S_0 with center at the origin, the circles normal to Σ and which cut the sphere in diametrically opposite points, or orthogonally, admit an infinity of normal surfaces which are in every case surfaces of the parabolic type and the axes of these circles are cut orthogonally by a family of surfaces of the elliptic type in the former case and of the hyperbolic type in the latter case; when S_0 reduces to a point the surfaces normal to the axes are of the parabolic type.* Hence with every surface of the parabolic type we can adjoin a surface of the elliptic, hyperbolic or parabolic type after the above manner.

Let the given parabolic surface Σ be defined by (85). Then the rectangular coordinates of the adjoined surface are[‡]

$$(94)\quad x=\left[\tfrac{1}{2}\left\{e^{\alpha}-(\beta^2+\varkappa)e^{-\alpha}\right\}\cos\theta+\beta\sin\theta\right]X_1+\left[\tfrac{1}{2}\left\{e^{\alpha}-(\beta^2+\varkappa)e^{-\alpha}\right\}\sin\theta-\beta\cos\theta\right]X_2-t\,X_3,$$

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[*] Annali, ser. 3, vol. 24, p. 376.

[†] Annali, vol. 24, p. 367.

[‡] Annali, ser. 3, vol. 24, p. 368.

and similarly for y and z, where t is given by

$$(95)\qquad \begin{aligned} \frac{\partial t}{\partial u_1} &= [\tfrac{1}{2}\{e^{\alpha}-(\beta^2+\varkappa)e^{-\alpha}\}\cos\theta+\beta\sin\theta]\sin\omega, \\ \frac{\partial t}{\partial u_2} &= [\tfrac{1}{2}\{e^{\alpha}-(\beta^2+\varkappa)e^{-\alpha}\}\sin\theta-\beta\cos\theta]\cos\omega, \end{aligned}$$

where $\varkappa$ being constant which is positive or negative according as the surface is of the elliptic or hyberbolic type, and vanishes when it is of the parabolic type, and ω is given by (34'). For this case we have

$$(96)\qquad \begin{aligned} H_1 &= \tfrac{1}{2}\{e^{\alpha}+(\beta^2+\varkappa)e^{-\alpha}\}\cos\omega+t\sin\omega, \\ H_2 &= \tfrac{1}{2}\{e^{\alpha}+(\beta^2+\varkappa)e^{-\alpha}\}\sin\omega-t\cos\omega. \end{aligned}$$

We consider now the parabolic system whose linear element is given by (89), the function α being determined to within an additive constant by (84) and (93) and β directly by (87), and determine the conditions under which the surfaces associated with these parabolic surfaces in the manner mentioned above form a family of Lame. These adjoined surfaces defined by (94) have the same spherical representation as the pseudospherical surfaces (90) and consequently ω satisfies all of the equations (8), (18), and (19). The expressions for H_1 and H_2 given by (96) satisfy equations (7) and the remaining condition (12) can be reduced to

$$(97)\qquad \tfrac{1}{2}\{e^{\alpha}-(\beta^2+\varkappa)e^{-\alpha}\}\frac{\partial\alpha}{\partial u_3}+\beta e^{-\alpha}\frac{\partial\beta}{\partial u_3}+e^{-\alpha}\frac{\partial\varkappa}{\partial u_3}+t\frac{\partial\omega}{\partial u_3}=0.$$

When the value for t obtained from this equation is substituted in (95), the latter become in consequence of (87), (92), and (96)

$$\begin{aligned} \left(\cos\theta\frac{\partial\omega}{\partial u_3}+\frac{1}{\cos\omega}\frac{\partial^2\omega}{\partial u_1\partial u_3}\right)\frac{\partial\varkappa}{\partial u_3} &= 0, \\ \left(\sin\theta\frac{\partial\omega}{\partial u_3}+\frac{1}{\sin\omega}\frac{\partial^2\omega}{\partial u_2\partial u_3}\right)\frac{\partial\varkappa}{\partial u_3} &= 0. \end{aligned}$$

The quantities in the parentheses cannot vanish unless the original pseudospherical system has constant curvature, which is contrary to hypothesis. Hence $\varkappa$ is a constant, and when α and β are determined in the manner remarked above there is a system with the linear element

$$(98)\qquad \begin{aligned} ds^2 = {} & [\tfrac{1}{2}\{e^{\alpha}+(\beta^2+\varkappa)e^{-\alpha}\}\cos\omega+t\sin\omega]^2\,du_1^2 \\ & +[\tfrac{1}{2}\{e^{\alpha}+(\beta^2+\varkappa)e^{-\alpha}\}\sin\omega-t\cos\omega]^2\,du_2^2 \\ & +\left[\tfrac{1}{2}\{e^{\alpha}+(\beta^2+\varkappa)e^{-\alpha}\}-\frac{\partial t}{\partial u_3}\right]^2 du_3^2, \end{aligned}$$

where t is given directly by (97) and all the surfaces of the family are of the elleptic, parabolic, or hyperbolic types according as $\varkappa$ is positive, zero, or negative.

In our previous discussion of surfaces of Bianchi it was shown that the complementary transform of the surface, defined by (94), by means of the function θ appearing in these expressions leads to the surface (85) where α, β, θ are the same as in (94). It follows therefore, that the parabolic system with the linear element (89) is one of the two A-systems which are the complimentary transforms of the system (98). The other transformed system also is parabolic, as we have shown in considering individual surfaces, and the linear elements of the transforms are given by

$$\begin{aligned} ds^2 = & \left[\tfrac{1}{2}\left\{e^\alpha + (\beta^2 + \varkappa)e^{-\alpha}\right\}\cos\theta + \beta\sin\theta + \tfrac{1}{2}\left\{e^\alpha + (\beta^2 + \varkappa)e^{-\alpha}\right\}\cos\theta'\right] du_1^2 \\ & + \left[\tfrac{1}{2}\left\{e^\alpha - (\beta^2 + \varkappa)e^{-\alpha}\right\}\sin\theta - \beta\cos\theta + \tfrac{1}{2}\left\{e^\alpha + (\beta^2 + \varkappa)e^{-\alpha}\right\}\sin\theta'\right] du_2^2, \end{aligned}$$

where θ' denotes the second solution of (91).

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